Explicit Inversive Pseudorandom Number Generators

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Contents

As a rule, random number generators are fragile and need to be treated with respect. It's difficult to be sure that a particular generator is good without an enormous amount of effort in the various statistical test.

The moral is: do your best to use a good generator, based on the mathematical analysis and the experience of others; just to be sure, examine the numbers to make sure that they "look" random; if anything goes wrong, blame the random-number generator !

– Robert Sedgewick, in "*Algorithms*" (Second Edition, 1988)

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Chapter 1

Introduction

The purpose of this thesis is to discuss the implementation of the explicit inversive congruential generator (EICG) and the properties of the resulting pseudorandom numbers. But before we delve into the details of the implementation or the theoretical and empirical results we will take a closer look at the basic concept of pseudo-random numbers.

What do we mean when we talk about *pseudo*random numbers (PRN) ? And for what purpose do we devise such elaborate means to artificially generate megabytes of digital noise ?

1.1 What do we need Pseudo-Random Numbers for ?

To the uninitiated, all this pseudo-random numbers "business" seems to have no serious applications. Everybody will come up with computer games as a field where pseudo-random numbers are used to make the behaviour of the computer less predictable. Steering the movements of some on-screen monster does not require a high standard of randomness, almost any algorithm will suffice, provided it is easy to implement and does not cost too much computing resources.

Another domain where we need PRN is wherever we need to model a more or less random phenomenon of the real world. The simulation of a roulette table or other forms of lottery might still be in the area of non-serious application, but here the defects of the generator start to be an issue. Imagine this scenario: you try to develop a winning strategy for blackjack and use a simulation to test your algorithm. Any correlation between statistical defects and the strategy will lead to a skewed result and may even change the sign of the expected outcome. Playing that strategy in a real casino might cost you dearly. Thus it is important to make

the choice of the generator an issue even in such non-scientific applications.

Simulation of random events is far from being limited to gambling, a significant percentage of all simulations of natural phenomena contains a random component. Whether that may be quantum effects, rainfall on a certain area, Brownian motion, absorption pattern, bifurcation of tree roots, failure of technical components, solar activity ..., in all cases we know at best the statistical properties of an event. An analytical solution of the given problem based on probabilities is often not possible. Thus one has to resort to stochastic simulation (see [3, 47, 74]) where one calculates the result of the overall simulation by choosing possible outcomes of the underlying random events according to their respective probability. Doing this a number of times should provide enough samples of the outcome to estimate the probability of each possible result. Needless to say, the selection of the realisations of the underlying random events is crucial to the correctness of the whole calculation. Since this selection is done with the aid of PRN, their quality plays an important role in the whole process.

Finding the use of PRN in stochastic simulation is not that surprising, but finding them in algorithms for such mundane tasks like integration might need some more explanation. Numerical integration is a common problem in a great deal of real world problems. A big battery of algorithms (trapezoid method, Simpson's method, spline quadrature, adaptive quadrature, Runge-Kutta, ...) was developed to minimize the calculation costs while increasing the accuracy of the result. All these methods scale very badly with the dimension of the integral, so a completely different approach is more appropriate there. The Monte Carlo method (see [3, 47, 77, 83]) uses randomly selected samples of the function to estimate the integral. Provided we know more about the behaviour of the function (i.e. its total variation) and the distribution of the actual samples used (as measured by their discrepancy), the inequality of Koksma-Hlawka (See page 44, [50, 69]) will give an error bound for this method. Since it is generally not possible to calculate the exact value of the discrepancy of the random numbers used for the integration, this error bound will only be a probabilistic one. In order to get a deterministic one, the random numbers, which determine which samples of the function will be evaluated, are replaced by numbers for which the order of the discrepancy is known. This turns the Monte Carlo method into the *Quasi*-Monte Carlo method.

One way to get such good point sets is to explicitly construct them with that goal in mind (See [69] for a discussion of (t, m, s)-nets and other methods.) or use a PRNG for which an upper bound on the discrepancy is known. This is one of the reasons why we will take a close look at this quantity in Section 3.3. Not all applications of Quasi-Monte Carlo integration are labeled as such, as the basic algorithm can be regarded as a simple heuristic. For example, distributed ray tracing [31, p. 788] uses a set of randomly distributed rays to implement

spatial and temporal antialiasing which amounts to an integration over both the time-frame of the picture and the pixel's spatial extension.

Non-deterministic algorithms often use pseudorandom numbers, too. These algorithms are used to tackle problems for which a deterministic solution takes too much time. Although they cannot guarantee success, they promise to find the solution (or a sub-optimal one) within reasonable time. Examples for this kind of algorithm are Pollard's rho heuristic [8, p. 844] for integer factorization, the Rabin-Miller primality test [8, p. 839], Simulated Annealing [30], and Threshold Accepting [11].

Other algorithms use pseudorandom numbers for a different purpose. Instead of using them directly for solving the problem they are used to *randomize* the problem (or the algorithm) in order to avoid running into the same worst-case behaviour again and again. See [8, p. 161] for an explanation of the rationale behind randomized quick-sort.

Some cryptographic algorithms and protocols require a good source of random numbers, too. Stream ciphers [76, p. 168f], for example, use the output of a PRNG (termed keystream generator) to encrypt the plaintext. The security of this cipher depends largely on the statistical quality of the keystream. Any regularities of the PRNG can be used by the attacker to predict the next bits and thus crack the code. No other domain of PRNG applications has such a high demand on the "randomness" of the generated PRN. Algorithms which are good enough for stochastic simulations are typically way too predictable to be useful as a keystream generator for a stream cipher. Thus the field of cryptographically secure PRNG has amazingly little in common with the study of PRNG for stochastic simulation on which we will focus in this thesis. Information on cryptographically secure pseudorandom numbers can be found in [51], [81], and [76].

Another application of pseudo-random numbers in the field of cryptology is providing the "random" numbers needed for a variety of cryptographic protocols. A well known example are the session keys generated for each transaction in hybrid cryptosystems. As the recent debacle involving the Netscape Navigator [32, 68] has shown, one must be very careful not to use a simple PRNG for this task. Since this is more a matter of how to get the entropy needed for non-predictability than one of analysing the properties of sequences of PRN we will not elaborate on this subject in this thesis.

1.2 Criteria for PRN Generator Selection

Now that we know a bit about the various applications of PRN, let's try to formulate a few criteria for the selection of a *good* PRN generation algorithm. As

we will see later it is crucial for the selection of the right PRNG to keep an eye on the application of the PRN.

1.2.1 Reproducibility

This criterion may sound strange at first sight, since reproducibility contradicts the intuitive notion of randomness, and indeed, *real* random number generators are extremely unlikely to ever repeat their output. So what are the advantages of a generator which will produce the same sequence of pseudorandom numbers when fed with the same parameters ? Once again, we have to turn to the application of which the generator is a component. In the case of a stochastic simulation the benefit is twofold:

- As a scientific experiment, *it should be possible to redo* the same calculations under the same conditions. This ensures that an independent verification of the obtained result is possible.
- *Debugging and verifying* the simulation program is greatly helped by the possibility of replaying the calculation. Otherwise it may not be possible to determine if an unexpected outcome is caused by a systematic error in the simulation setup, or whether it is just a statistic fluke.

In some areas, for example stream ciphers, reproducibility is a key requirement for the application. Only very few applications, most of them in the area of cryptography, do actually benefit from the use of non-reproducible PRN.

1.2.2 Statistical properties

It is clear that when we want to simulate a random variable with a PRNG, then the output of the generator should model as closely as possible the expected behaviour of instances of the random variable. If a simulation of a dice generates a 7 or strongly favors the 6 we will not accept the generator. Other deviations from the desired behaviour, e.g. correlations, are harder to detect, and methods for systematically testing generators for such deficiencies have been the subject of considerable mathematical work [49, 27, 53, 84], including some parts of this thesis.

As we will see later, proving that a generator really has all the statistical properties a real random number generator is supposed to have, is not possible. So all we can do is to establish faith in the generator by testing it for some properties.

1.2.3 Empirical Test Results

Empirical testing usually involves using the PRN for a stochastic simulation with a known result. If the computed results contradict the expected ones, the generator will be dismissed as not suitable for that kind of stochastic simulation. A passed test will increase the faith that this generator will yield correct results in real world problems. We will examine the significance of empirical test results later in greater details.

A large battery of such tests was developed over the years, from the well known tests of Knuth [49] and Marsaglia [65] to recent additions like the weighted spectral test [39, 40, 45, 43]. See [53, §3.5.] for further references on testing pseudorandom number generators.

1.2.4 Possibility of Theoretical Analysis

In order to make analytical investigations possible, most modern PRNG are defined in quite simple mathematical terms. It is a tradeoff: The simpler the algorithm, the easier it will be to prove statements concerning the quality of the generated numbers. On the other hand, a convoluted algorithm appeals to the intuition. History has shown [49, 73] that quite a few people could not resist the temptation to build generators based on doing obscure transformations on numbers stored in computers. Empirical analysis has shown that the quality of such generators are often abysmal.

1.2.5 Results of Theoretical Analysis

Doing empirical studies on the properties of a PRNG is always possible, but deriving properties of the generator output by pure mathematical study has a lot of advantages. Whereas an empirical test can only cover one specific set of parameters of a generator, it is sometime possible to make analytically proven statements on the properties of PRN generated by a certain generator *regardless of the parameters used*. In the same vein, an empirical test on a specific part of the generator's output, say the first billion numbers, may give us confidence on the behaviour of the next billion numbers, but cannot offer any guarantee that they will be equally good. Analytical results fall in the following categories:

- **Basic parameter selection**

  For most generators not all possible parameters will result in a functional generator. A typical question is that of gaining the largest possible period length. For the LCG[1] this is just a set of simple conditions, for the ICG

[1] See 1.3 for the definition of this and other generators.

it involves finding IMP polynomials [6, 29, 44, 41, 42, 24]. For compound generators to work it is also necessary to obey certain analytically derived constraints.

- **Properties of the resulting PRN**

  For some generators it is possible to derive statements on some aspects of the output. The well-known fact that tuples of LCG generated numbers form lattices (see page 37) is one example.

- **Estimates and bounds**

  Especially the discrepancy has been the subject of analytical study. There are numerous estimates and bounds for various generators.

1.2.6 Efficiency

With all the mathematical discussions about the merits of PRN generated by a new algorithm one should not forget the fact that we need to actually implement this algorithm on a real computer. There are a few things which should be noted here:

- **Implementation costs**

  Implementing (i.e. programming) an algorithm is usually a one-time investment of effort. Once the code is there, integrating it into a larger project is more or less trivial. What are the difficulties in implementing a typical pseudorandom number generation algorithm ? As we will see later in Chapter 5, the main problem lies in the handling of large integers and performing standard mathematical operations like addition and multiplication on them. For inversive generators finding the multiplicative inverse in $\mathbb{Z}_p^*$ is a required operation, too.

- **Computational costs**

  The execution of any algorithm requires both CPU and memory resources. Typical PRNG (EICG, LCG, ICG, ...) have only very small memory requirements. The code is very compact and the state information does only require a few bytes.

  As far as CPU consumption is concerned, a PC with Intel 486DX2-66 processor is capable of executing the EICG algorithm about 70000 times in a second. The author's implementation of the LCG runs at about 400000 PRN per second. The highly optimized system pseudorandom number generator runs at over 700000 calls per second. These numbers are only provided to give a rough feeling for the speed of the algorithms when using a modulus in the range of 2^{31}.

- **Implications for the overall running time**

  As the generation of the PRN is usually performed on demand on the same computer as the stochastic simulation for which they are used, they compete for the same resources. The total running time for the simulation can be described as the sum of the time used for the PRN generation plus the time used for doing the actual calculations. The latter often dominates the former, thus it does not make sense to try to gain overall speed by sacrificing quality in the PRN algorithm.

1.2.7 Practical Aspects

After implementing the algorithm one has to find good parameters for that generator, too. Fortunately, for some common PRNG tables containing suitable parameters have been published [28, 42, 2, 82], so there is no need to reinvent the wheel there. Other generators like the EICG are known to be rather insensitive to the choice of the parameters.

Another aspect is the possibility to generate independent streams of pseudorandom numbers. Such streams are needed for parallel or vectorized computing. See [54, §8], [1], [12], and [35] for more information on this topic.

1.3 Important Types of PRN Generators

There is no shortage on proposed pseudorandom number generation algorithms. Every year new ideas on this topic are published, but only if the resulting PRN have been subject to intensive *theoretical and empirical study* the generator might have a chance to get used in a real world problem. As it is often the case with competing inventions, an objective technological superiority does not immediately lead to market domination. Whether the generator is included in standard programming libraries seems to be much more important than any published results on the distribution properties of the numbers. A classic example is the now infamous RANDU generator which was included in IBM's Fortran library and features an extremely poor distribution of triples composed of subsequent numbers.

The following list introduces some of the most commonly used generators as well as the inversive generators on which we will focus in this thesis. More complete surveys on the current menagerie of PRNG can be found in [69, 71, 54].

In the following M denotes a positive integer (termed modulus) and $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$ represents the system of all residues modulo M. With the addition and multiplication modulo M the set $\mathbb{Z}_M$ acquires the algebraic structure

of a finite ring. If the context makes it clear that we operate in the ring $(\mathbb{Z}_M, +, \cdot)$ we will omit the trailing "mod".

1.3.1 The Linear Congruential Generator

Definition 1.1 *Let $a, b, y_0 \in \mathbb{Z}_M$. The* linear congruential generator *(abbreviated as "LCG") with parameters M, a, b, and y_0 defines a sequence $(y_n)_{n\geq 0}$ in $\mathbb{Z}_M$ by*

$$y_n := a \cdot y_{n-1} + b \qquad (n > 0)$$

and a sequence $(x_n)_{n\geq 0}$ of pseudorandom numbers in $[0, 1[$ by

$$x_n := \frac{y_n}{M} \qquad (n \geq 0).$$

As the sequence $\mathrm{lcg}(p, a, b, y_0) = (y_n)_{n\geq 0}$ is defined by a recursion of order one on a finite set it must be periodic. The longest possible period length is M in the case of $b \neq 0$ and $M - 1$ in the case of $b = 0$. The necessary conditions for achieving these period lengths are well known. [69, p. 169]

The LCG is very popular. Its implementation is quite simple, especially if M is chosen as 2 to the power of bits per native word of the computer (e.g. 2^{32}) which reduces the modulo operations to just ignoring the overflow. Due to its simplicity and popularity the LCG has been subjected to intensive analytical and empirical examination. The quality of the resulting PRN depends very much on the choice of the parameters M, a, and b. Fortunately, tables containing good parameters have been published, see [28, 26, 52].

The output of a LCG shows a strong intrinsic structure ([64], see also p. 37). A number of modifications were proposed to improve the quality of the generator. One approach is to extend the recursion to higher orders by making y_n a function of $y_{n-1}, \ldots, y_{n-r}$. Other proposals modify the function which describes the recursion. As the name says, the LCG uses the *linear* function $f(y_n) = a \cdot y_{n-1} + b \pmod{M}$ to calculate y_n from y_{n-1}. If we replace f by an arbitrary function, we refer to the resulting PRNG as a *general first-order congruential generator* [69, p. 177]. In order to guarantee maximal period length, the function f must be carefully selected. For example, the *quadratic congruential method*, as proposed by Knuth in [49, §3.2.2] uses a polynomial of degree 2 as the recursion and a power of 2 as the modulus. See [69, p. 181f] for the conditions on the parameters and analytical investigation on the resulting PRN.

1.3.2 Shift-register Generators

Shift-register generators differ from standard linear congruential generators in two respects. First, they use a higher-order linear recursion of the form

$$y_{n+k} \equiv \sum_{h=0}^{k-1} a_h y_{n+h} \pmod{M} \quad (n \geq 0) \tag{1.1}$$

where $M \geq 2$ is the modulus, $k \geq 1$ is the order of the recursion and $a_0, \ldots, a_{k-1}$ are elements of $\mathbb{Z}_M$. Second, instead of just scaling the y_n to the unity interval to get the pseudorandom numbers, the x_n are calculated from a block of consecutive values $y_n, \ldots, y_{n+m}$. Thus it is no longer necessary to use a large modulus to get a decent resolution of the resulting PRN. In order to simplify and optimize the implementation of recursion, the common choice of M is the prime 2. On a L-bit computer this allows the grouping of L steps into one operation.

Two techniques for the transformation of the sequence $(y_n)_{n\geq 0}$ into a sequence of pseudorandom numbers in $[0,1[$ are commonly used: The *digital multistep method* puts

$$x_n = \sum_{j=1}^{m} y_{mn+j-1}\, p^{-j} \ \in [0,1[\quad (n \geq 0). \tag{1.2}$$

The *Tausworthe* generator [80] is a special case of this method.

More popular is the *generalized feedback shift-register method (GSFR)* which can take advantage of the above mentioned blocking of L bits if $h_1, \ldots, h_m \geq 0$ are selected suitably:

$$x_n = \sum_{j=1}^{m} y_{n+h_j}\, p^{-j} \ \in [0,1[\quad (n \geq 0). \tag{1.3}$$

If the parameters are carefully selected the period length will in both cases be $\mathrm{per}(x_n) = p^k - 1$.

Shift register pseudorandom numbers have the advantage of a fast generation algorithm and a period length independent of the limitations of the integers used for the calculation. See [69, Chapter 9] and [54] for a discussion on the properties of shift register pseudorandom numbers.

1.3.3 The Inversive Congruential Generator

A promising modification of the LCG was proposed by Eichenauer and Lehn in [14]. We will only consider the case of a prime modulus $p = M$ here. It involves the operation of modular inversion in $\mathbb{Z}_p$ which we will denote by an overline ($\overline{c}$).

$$\overline{c} = \begin{cases} c^{-1} & \text{for } c \in \mathbb{Z}_p, c \neq 0 \\ 0 & \text{for } c = 0 \end{cases} \tag{1.4}$$

The restriction to prime moduli guarantees the unique existence of an inversive element in $\mathbb{Z}_p$. This definition implies $c\overline{c} \equiv 1 \pmod{p}$ for $c \neq 0$.

Definition 1.2 *Let p be a (large) prime and $a, b, y_0 \in \mathbb{Z}_p$. The* inversive congruential generator *(abbreviated as "ICG") with parameters p, a, b, and y_0 defines a sequence $(y_n)_{n\geq 0}$ in $\mathbb{Z}_p$ by*

$$y_n := a \cdot \overline{y_{n-1}} + b \qquad (n > 0)$$

and a sequence $(x_n)_{n\geq 0}$ of pseudorandom numbers in $[0, 1[$ by

$$x_n := \frac{y_n}{p} \qquad (n \geq 0).$$

Empirical as well as analytical investigations indicate that the output of an ICG is superior to the output of a LCG in several respects: longer usable sample sizes [84, 60], less correlations between consecutive numbers [71].

1.3.4 The Explicit Inversive Congruential Generator

Analytical calculations have led to the following observation: We can describe the generator as a function mapping n to y_n. This self-map $n \mapsto y_n$ in the finite field $\mathbb{Z}_p$ can be written as a uniquely defined polynomial g with degree $d < p$. If we demand the sequence $(y_n)_{n\geq 0}$ to have the maximal possible period length p, the polynomial g maps $\mathbb{Z}_p$ onto itself and thus must be a permutation polynomial, which is either linear ($d = 1$) or satisfies $3 \leq d \leq p - 2$ according to [63, Cor. 7.5]. It turns out that the degree d plays an important role in the analytical examination of the generator in a sense that a higher degree seems to indicate better distribution properties [69, Theorems 8.2, 8.3] (see p. 55). The theorem of Euler-Fermat tells us that evaluating c^{p-2} corresponds to the calculation of the multiplicative inverse. In this spirit, the definition[2] of the EICG seems quite natural:

Definition 1.3 *Let p be a (large) prime and $a, b, n_0 \in \mathbb{Z}_p$. The* explicit inversive congruential generator *(abbreviated as "EICG") with parameters p, a, b, and n_0 defines a sequence $(y_n)_{n\geq 0}$ in $\mathbb{Z}_p$ by*

$$y_n := \overline{a \cdot (n_0 + n) + b} \qquad (n \geq 0)$$

and a sequence $(x_n)_{n\geq 0}$ of pseudorandom numbers in $[0, 1[$ by

$$x_n := \frac{y_n}{p} \qquad (n \geq 0).$$

[2]The original definition does not include n_0.

As long as $a \neq 0$ this generator will always have period length p. Once again analytical and empirical investigations have shown that the output of this generator is superior to that of an LCG. This will be the generator on which we will focus our attention in this thesis. The other generators mainly serve as a reference against which the EICG must compete.

1.3.5 EICG Variants

Two variations of the basic explicit inversive congruential generator have been proposed. Both proposals substitute the prime modulus p with $M = 2^\omega \quad (\omega \geq 4)$. In the set $\mathbb{Z}_M$ we can define the modular inversion only for odd integers. This inversion is once again defined by $c\overline{c} = 1 \pmod{M}$ for all odd c.

Definition 1.4 *(Eichenauer-Herrmann and Ickstadt [21]) Let M be a power of 2, and $a, b, n_0 \in \mathbb{Z}_M$ with $a \equiv 2 \bmod 4$ and $b \equiv 1 \bmod 2$. The* explicit inversive congruential generator with power of two modulus *with parameters $p, a, b,$ and n_0 defines a sequence $(y_n)_{n\geq 0}$ in $\mathbb{Z}_M$ by*

$$y_n := \overline{a \cdot (n_0 + n) + b} \qquad (n \geq 0)$$

and a sequence $(x_n)_{n\geq 0}$ of pseudorandom numbers in $[0, 1[$ by

$$x_n := \frac{y_n}{p} \qquad (n \geq 0).$$

The conditions on a and b guarantee that the sequence $x_0, x_1, \ldots$ is purely periodic with period $M/2$. While powers of 2 as modulus have certain advantages for the implementation of the generator, all theoretical investigations [21, 15] on the quality of the resulting numbers have concluded that this generator is inferior to the original EICG.

In order to achieve a period length of M, Eichenauer-Herrmann [16] proposed the following generator:

Definition 1.5 *Let M be a power of 2, and $a, b, n_0 \in \mathbb{Z}_M$ with $a \equiv 2 \bmod 4$ and $b \equiv 1 \bmod 2$. The* modified explicit inversive congruential *with parameters $p, a, b,$ and n_0 defines a sequence $(y_n)_{n\geq 0}$ in $\mathbb{Z}_M$ by*

$$y_n := n \cdot \overline{a \cdot (n_0 + n) + b} \qquad (n \geq 0)$$

and a sequence $(x_n)_{n\geq 0}$ of pseudorandom numbers in $[0, 1[$ by

$$x_n := \frac{y_n}{p} \qquad (n \geq 0).$$

Although this modification does indeed increase the period length to M, the theoretically derived properties of the resulting numbers are still inferior to the original EICG.

1.3.6 Compound Techniques

An interesting *meta*-generator is the compound method. This is a very simple and effective way to combine several streams of PRN into one single sequence with (hopefully) superior properties. It works as follows: For $1 \leq j \leq r$ let $x_0^{(j)}, x_1^{(j)}, x_2^{(j)}, \ldots$ be a purely periodic sequence of pseudorandom numbers. Then we get the *compound* sequence $x_0, x_1, \ldots$ by

$$x_n = \sum_{j=1}^{r} x_n^{(j)} \bmod 1 \qquad \text{for } n = 0, 1, \ldots.$$

If the subsequences are purely periodic with distinct period $\mathrm{per}(x_n^{(j)}) = p_j$, then we have $\mathrm{per}(x_n) = \prod_{j=1}^{r} p_j$.

This compound method extends the well-known approach of Wichmann and Hill [87]. The properties of the resulting sequence has been subject to a number of publications; I refer to Niederreiter [71, 4.2] for all the references. Generally speaking, the compound method preserves the basic properties of the underlying generators.

Chapter 2

The Notion of "Randomness"

Examining what we mean by *random* numbers will help us to understand the difficulties in generating *pseudo-random* numbers and interpreting test results. We will look at how we all intuitively deal with supposedly random sequences, and touch upon the mathematical treatment of the subject. Regrettably, we will not be able to comprehensively cover this topic, thus we will focus on the subject of testing (finite) sequences of PRN.

2.1 Randomness by Intuition

First of all, we want to take a closer look at the intuitive notion of randomness. For one, we all intuitively assign probabilities to various events we encounter, from such mundane things like which side a dropped slice of bread will land on, every-day events like rainfall, the number of red traffic lights encountered, or friends met in the bus, to explicitly random events like the outcome of a dice or the weekly lottery.

But how do we come to the conclusion that one of these events is somehow random ? What are the criteria for that decision ? In some of the example above the decision is easy as we know about the process which leads to the outcome. Watching the dice being cast properly is a sure way to convince oneself that the outcome is indeed truly random. But how do we proceed when we cannot look behind the scenes, when the sequence of outcomes is the only information we have got ?

The human mind has remarkable capabilities to spot regularities in a sequence of events. If it fails to notice anything suspicious it will declare the sequence to be random.

Let's test this notion on the most widely used source of random numbers,

the dice. A dice is supposed to select one of the numbers $1, \ldots, 6$ in a fair and independent fashion each time it is cast. In the following list we will argue on the merits of a few possible outcomes.[1]

{2} If we cast the dice just once, each of the possible outcomes are equally likely. All sequences of length one are thus equally good.

{2, 4} There is nothing wrong with this sequence, too.

{6, 6} Casting doublets is not uncommon in the real world, thus they are not reason enough to doubt the fairness of the dice.

{1, 2, 3} A sequence of length 3 is too short to arouse any suspicion, too. Such simple patterns (runs up, runs down, only even numbers, only odd ones, only primes, ...) are actually quite likely to occur.

{1, 2, 3, 4, 5, 6, 1} With increasing length it is possible that a more clearly visible pattern emerges within the numbers. Would you accept such a sequence as generated by a real dice ? As these things still happen every now and then in real life, the common answer to this question seems to be "Yes, as long as this doesn't happen too often.".

{4, 2, 2, 2, 4, 2, 4} This sequence will give reason to doubt the fairness of the dice. Whether the perceived skew is reason enough to outright reject the sequence as random is a tricky question. After all, a real dice will show such irregularities from time to time, too.

{6, 6, 6, 6, 6, 6, 6, 6} Although theoretical a possible outcome of a dice, this sequence will probably not be accepted as such.

{6, 3, 2, 1, 2, 5, 4, 1, 6} At the first glance, this sequence looks quite random, but closer scrutiny shows that it features alternately even and odd numbers. As the probability for this to happen purely by chance is pretty small, one gets suspicious about the "randomness" of the sequence.

{1,3,4,6,5,3,2,1,1,3,4,6,5,5,4,2,1} This one looks inconspicuous, too. But if you draw the graph of the sequence you will notice a regularity: There are too many long ascending or descending subsequences. As these long "runs" should not appear that frequently in random sequences, this one does not seem to be random.

{1,2,5,4,3,6,6,2,4,1,3,5,5,2,3,4,1,6} What about this sequence ? Can you see any regularity in it ? Once again, a casual look will not find anything suspicious, the distribution of the numbers seems to be balanced, consecutive

[1] Imagine a friend casts the dice behind your back and announces the following outcomes. Would you believe him to have correctly reported the numbers when he announces the following lists ?

numbers do not have any special relation and there is nothing wrong with the runs, too. Actually, the numbers are balanced way too good as all six numbers appear before the first one gets repeated. In other words, the sequence consists of three permutations of $\{1, \ldots, 6\}$. If the equidistribution of the numbers is so perfect, the randomness of the sequence must be challenged.

{3,2,5,2,1,4,4,6,2,3,4,5,2,5,6,3,3,1,5,4} This sequence has a hidden regularity, too. In previous sequences we have looked for correlations between consecutive numbers. If we generalize this and take a closer look at numbers n steps apart, we have a more versatile tool for finding regularities. It turns out that $a_i + a_{i+10} = 7$ holds for all values of i. Such a *"long range" correlation* should not happen in random sequences.

Did you see the one big fault in this sequence of would-be random sequences ? We did not notice it because we looked only at single sequences. Can you find it now ?[2]

Let us summarize the arguments:

- *If the sequence has properties we do not expect to be present in random numbers, we get suspicious.*

  If we argue about the "randomness" of a given sequence we try to find reasons for rejecting it as random. There seems to be no way of asserting a sequence to be random, it is only the absence of arguments to the contrary that will lead to confidence in the sequence. The proper formulation in the language of statistics is the following: The *null hypothesis* is always to assume the sequence was indeed generated by a random process with well known statistical properties. As we will see later, it is not possible to reverse the problem and regard the non-randomness as the null hypothesis.

- *Longer sequences are easier to judge.*

  Short sequences are likely to contain some sort of perceived regularity, thus it is hard to reject such a sequence based on a suspicious pattern. If the sequence is long enough to check if the pattern continues to appear in it, one can try to determine if the pattern is part of some systematic fault or just coincidence.

- *There are a lot of ways a sequence can be suspicious.*

  Just when we thought we have found a sequence which does not exhibit the patterns we have found in all the previous faulty ones, it turns out that there is a different kind of regularity in it. Somehow this is just like the trick question for the first natural number without any special properties. If such

[2] Try summing each sequence up. It should be obvious then.

a number existed, the very fact would make it special, thus there can be no such number. We almost get the same feeling when we examine sequences for their non-conspicuousness. As there are so many ways a sequence can exhibit a pattern, a complete absence of patterns is just as conspicuous as any weak regularity.

Furthermore, it is worth pondering if there are not so many patterns that all sequences will exhibit one. We will take a closer mathematical look at this question later.

- *"Perfect randomness" is an oxymoron.*

  If the "random" sequence exhibits exactly the expected distribution this will cause suspicion, too. A random sequence is supposed to deviate from its distribution. The common measure for this is the *variation*. A sequence with a perfect distribution will fail to have the same variation a random sequence is supposed to have.

  As the variation can be viewed as just another test statistic, it, too, should vary in a certain way. From that point of view, a constant and perfect variation is just as suspicious as a constant and perfect distribution. This reasoning leads to the demand that not only the distribution of the numbers should be as wanted, but also that the empirical higher moments should be close to the values predicted by probability theory.

2.2 Formalizing the Intuitive Notion

Now that we have examined what we intuitively mean by saying "This sequence looks random." we can try to formalize this notion and develop a set of properties we want to check if we have to judge a sequence and its generating algorithm. The goal in this formalisation is to be able to delegate the testing to computer programs. As computers are known to be very bad at spotting patterns, it will not be an easy undertaking to find an algorithm which does as good as the human mind. We can only hope that all systematic faults in the sequence will eventually cause a suspicious behaviour of the sequence in a generic test.

In the following we abandon the dice as the example, and turn to uniformly distributed numbers in the interval $[0, 1[$.

Distribution:

The first step in testing a sequence is usually to test its distribution characteristics. That is, are the numbers equally spread over $[0, 1[$?

In order to test the (empirical) distribution one partitions the interval $[0, 1[$ in sets A_i and compares the number of hits in each set to the size (measure) of that interval.

In the discrete case this can be done by simply counting how often each possible value appears in the sequence. If the counts differ significantly, the distribution property of the sequence is inadequate.

In order to keep the problem manageable in the case of a huge number of possible outcomes and in the continuous case, the bins (i.e. the A_i) used for counting will cover more than one outcome.

The layout of the partition is a crucial part of the test: If the A_i are simple intervals the test will measure the overall distribution of the sequence. But the A_i could be the union of a set of small intervals, in which case the test targets irregularities in the fine structure of the sequence.

Once we have finished the counting process we need some mathematically justified criteria for interpreting the difference between the number of hits in each A_i and the expected count. There are a number of possible algorithms for this, the most popular of which are the χ^2-test and the Kolmogorov-Smirnov test (often abbreviated as KS-test). The former uses a test statistic based on the difference between expected and actual count in each bin, whereas the latter compares the empirical distribution function of the counts to the expected one.

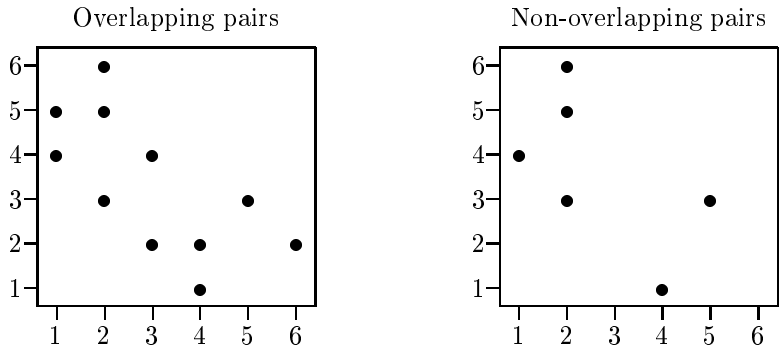
Correlations:

It should be clear that any numbers in a deterministically generated and thus reproducible sequence are trivially correlated. Therefore it makes no sense to look for such intricate dependencies like the generation rule in the sequence. We will restrict our search to much simpler correlations, which makes additional sense because that will be the only kind of correlations we can hope to find with the limited capabilities of a computer program. There are two approaches to this:

Tests for *special correlations* check if the sequence exhibits a given kind of regularity. An often used example is the *run-test* which measures the frequency of ascending or decreasing parts in the sequences. The distribution of these *runs* in random sequences is known, making it possible to judge the sequence with respect to this type of correlation.

The *serial test* is a more general way of examining a sequence. It transforms the problem of testing for correlation to the problem of testing for equidistribution by looking at tuples composed of elements from the sequence. The size of each tuple is called the *dimension* $s \geq 2$ of the test. Common tests use either *overlapping tuples* defined as $\mathbf{x}_n := (x_n, x_{n+1}, \ldots, x_{n+s-1})$, or *non-overlapping tuples* defined as $\mathbf{x}_n := (x_{sn}, x_{sn+1}, \ldots, x_{sn+s-1})$. If there are no correlations in the original sequence the s-tuples are equidistributed in the unit cube of dimension s, which can be checked using the techniques outlined above.

To illustrate this, let us examine the sequence $\{1, 4, 2, 6, 2, 3, 4, 1, 5, 3, 2, 5\}$ with the serial test of dimension 2.



As you can see, the fact that large and small numbers alternate causes a significant deviation from the equidistribution of the points.

For a number of generators it is possible to derive analytical bounds for the deviation from the equidistribution of s-dimensional tuples as measured by the discrepancy.

If one does not restrict oneself to form tuples out of consecutive numbers, the resulting test will be able to find more subtle kinds of correlations without resorting to high dimensions s. While this modification hardly changes the empirical testing, only in the case of the EICG analytical bounds have been derived for this generalized serial test.

2.3 Randomness in Mathematical Terms

Now that we have clarified the intuitive understanding of the concept of testing pseudorandom sequences, we will turn to the mathematical treatment of the subject. Rather than providing a full scale discussion of the mathematical objects and formalisms involved, which would exceed the scope of this thesis, we want to present an introduction targeted at the mathematical layman. Our aim in this section is to introduce as much of relevant concepts as is necessary to be able to explain the problems one faces when testing pseudorandom numbers and comparing PRNG. We refer to [84] for an in-depth discussion.

There is more than one mathematical approach to this topic. The following list tries to introduce the different viewpoints and gives references for further reading.

- **Number-theoretic approach.**

  In our context, this branch of mathematics focuses on the **equidistribution** of a sequence of numbers.

Various measures for the quality of the equidistribution were developed over the years, of all these numbers, the *discrepancy* is the most common.

Theorems on the equidistribution usually deal with infinite sequences, thus they are not particularly useful in conjecture with finite (or periodic) sequences. For example, equidistribution of a sequence can be defined in terms of the discrepancy in the following way:

$$(x_i)_{i=0}^{\infty} \quad \text{equidistributed} \quad \Longleftrightarrow \quad \lim_{N\to\infty} D_N(x_1, \ldots, x_N) = 0$$

See [50, 69, 71] for further reading.

- **Kolmogorov complexity and information theory.**

  This approach targets the complexity and information content of the sequence in question. One of the possible measurements is the minimal size of a computer a program (or a Turing machine) which can reproduce the sequence. In the optimal case, the program code will have to explicitly contain the sequence in order to print it. Any possible shortcuts the program can use (like exploiting dependencies) will be a measure for the lack of randomness of the sequence.

  Since all our sequences are generated by short programs, they a-priori fail this test. Thus we will not consider this notion in our tests.

  A similar approach is to focus on the amount of information contained in the sequence. If the entropy is high enough, we will accept the sequence as a good approximation of random numbers. Another way to express this notion is to state that the sequence is not compressible.

  Testing whether a sequence is compressible is not easy since all common implementations cannot achieve the theoretically possible compression. Only really bad PRN can be eliminated with programs like `gzip` or `compress`. Extending the capabilities of these programs (for example enlarging the range of the pattern search in `gzip`) might be a way to get a workable test. As far as we know, nobody has tried this yet.

  For larger sequences, the distinction between these ideas start to blur, as the size of the information needed to transform one representation into the other becomes irrelevant.

  See Lagarias [51], Ming/Vitány [66] and Chaitin [5].

- **Cryptographic considerations.**

  A sequence of PRN can be used to construct a *stream cipher*. If "true random" numbers are used, this cipher is called the *one-time pad* and is provably secure. So it is natural to ask what properties the PRN must have to achieve a good level of security.

According to Rueppel [75] there are several approaches to the construction of a secure stream cipher: The *information-theoretic approach* considers the possibility in principle to derive the seed (i.e. the key) from an observation of the PRN, the *system-theoretic approach* tries to make breaking the cipher at least as hard as solving known "hard" problems like factoring or the discrete logarithm. The *complexity-theoretic approach* tries to make sure that the amount of work needed to break the cipher is of non-polynomial complexity, *randomized stream ciphers* increase the magnitude of the code-breaker's problem by utilizing a public pool of random numbers.

See [76, 75] for a discussion of these ideas.

- **Statistical approach.**

  The basic idea of statistical testing can be summarized as follows: From a sample of supposedly random numbers a function called *test statistic* is calculated. As the distribution of this function is known for the case of *real* random numbers (otherwise the test does not make sense), one can determine which kind of results are extremely unlikely to occur. Typically this is formulated as intervals in the domain of the test statistic. These intervals (usually called *critical region*) are selected in a way that the probability that real random numbers lead to a test statistic there is smaller than the *level of significance* (usually 0.05, 0.01 or 0.005). If now for a sample of PRNG the test statistic falls into the critical region the common inference is to reject the sample.

  See [49, 4, 53] for further reading.

All common tests rely on the idea of statistical testing. In the following we will try to elaborate on the motivation behind these tests, their mathematical foundation, their power and limitations, and how to interpret their results.

2.3.1 Random Variables and Probability

First of all, let us take a closer look at what we want to simulate. Our target are sequences of random numbers, which are realisations of a sequence of independent, uniformly distributed random variables.

Random variables (RVs) are one of the main building blocks in probability theory. They are used to assign each possible outcome (or, to be more exact, each reasonable set of outcomes) of an experiment a real number which is interpreted as the *probability* of this outcome.

But strictly speaking, the mathematical concept of RVs does not explicitly reflect our intuitive ideas about randomness of events, on the contrary: RVs are just simple, ordinary functions. One is tempted to ascribe mythical powers to

RVs, like the ability to *randomly* select one of a set of possible events. This is not true, they only *describe* certain aspects of an *idealized* system which flips the metaphorical coin.

So where is the link between the mathematical world of RVs and the real life world of roulette tables ? Unfortunately there is none for single events. Even if a RV does in fact model a real world event, hardly any conclusions can be made about the outcome of the next single event. Even such unlikely events as winning the jackpot in a lottery do happen every now and then, and most people are not deterred by the extremely bad odds from playing every week. On the other hand some people are scared of travelling by plane because the probability of a safe flight is marginally less than one. In both cases our experience tells us that the probability alone cannot predict the next outcome.

But even such pretty definite sounding statements like "this event will occur with probability 1" cannot guarantee the outcome of an event. More insight into measure theory will tell us why such strange things can happen. For example, the probability that the next realisation of an $U([0,1[)$-distributed random variable will be a rational number is zero. This does not stop the real world from delivering one of the infinite number of rational numbers, thus rendering the statement "This experiment will only return irrational numbers" incorrect.

We have seen that a RV cannot make concrete statements about a single outcome, so we might ask what statements about outcomes it can make at all. One way to formulate the meaning of probability is the following: [84, p. 10]

> The probability assigned to an event expresses the *expected* average rate of occurrences of the event in an *unlinked* sequence of experiments.

We need to elaborate on two aspects of this definition as they are not as strict and unambiguous as commonly demanded from a good definition.

First, what do we mean by "expected" ? That seems to indicate that probability cannot be an intrinsic property on an event. There is no mathematically satisfying way to assign a probability to an event based on a (finite[3]) set of measurements, as it is extremely unlikely that another set of experiments will result in the same value. The common way out is to make assumptions about some parts of the experiment, like the Laplace assumption which assigns the same probability to all underlying events. These assumptions are based on a mental model of that event which includes a theory on how often something should occur. It is the mathematician, the physicist or just some observer who forms a mental model based on experiences or consideration. Such simplifying mental models of the real world are ubiquitous as they provide an essential simplification in the way we view the world. Other such simplifications include the concept of

[3]It is possible to prove convergence as the number of measurements increases. [84, p. 20]

rigid bodies, fluids, or gases which are abstractions of "a bunch of molecules tied together by various forces". Just as the laws of leverage rely in their formulation on the concept of forces and of rigid bodies the laws of chance depend on the concept of probability assigned to events.

The other critical word in the above definition is "unlinked". By unliked we mean that the outcome of one experiment does not influence the outcome of any subsequent experiment. Common examples for unlinked experiments include drawing balls from an urn (*with* putting them back in !), casting a dice, or the roulette wheel. Please note that in all these examples there is a connection between two successive experiments as the first one *does* influences the second. It is a conscious decision by the observer that the re-shuffling of the balls in the urn caused by the first experiment does not affect the *probability* in the second one. This sounds almost like a paradox, as the re-shuffling surely does effect the *outcome*. But remember, just above we noted that the probability of an event does not determine the next outcome at all, so there is not contradiction here.

We have to be careful with sequences of PRN and their relation to independent random variables, too. The concept of *independence* is based on the concept of distributions. As we cannot ascribe distributions to numbers, we cannot use the term "independence" for sequences of PRN. We will use the word *correlations* to refer to any unwanted relationship between elements in the sequence.

2.3.2 Testing

As described above, the theory of random variables and probability tries to model aspects of the physical world. The fundamental principles of science demand justification in form of experiments for all such theories. For typical physical models such experiments are usually easy to set up and follow the same scheme of comparing an expected (calculated) result to the measurements of the actual physical event. If they differ more than inaccuracies in the measurements would allow, the theory is proven to be wrong. Philosophy of Science tells us that it is *impossible* to positively prove a theory.

Do the same principles hold for conjectures in the field of probability, too ? Unfortunately, they do not. Let us illustrate this with an example:

As a theory to test we might take the assumption that a given coin is fair, meaning that the probability it lands with the heads side up is $1/2$. How might an experiment designed to test this hypothesis look like ? Surely it will involve throwing the coin a number of times and then comparing the result to the prediction. Calculating the prediction based on the theory is simple, unfortunately the prediction assigns each possible outcome a positive probability. Thus regardless of the behaviour of the coin the result is consistent with the theory, as we cannot

rule out the measured result. If we have no way to reject a theory, we have to find a different set of criteria according to which we can justify theories.

The common way out is **statistical testing**. It should be clear that statistical testing can never be as strict as testing in other areas. It is a heuristic approach to the problem. As such, it relies on the good judgement of the tester and is *not* objective. But before we elaborate on the shortcomings of statistical testing let us summarize the basic procedure again, already using the test for randomness as the example.

1. The first step is to formulate the hypotheses. In our case the *null hypothesis* H_0 states that the source of our pseudorandom numbers can be modeled as a RV with distribution $U([0,1[)$.

   The *alternative hypothesis* H_1 states that H_0 is not true.

2. We define a function called test statistic which maps the result of an experiment into some mathematical domain. Typical test statistics for testing uniform PRN are the number of runs, the discrepancy, the χ^2-statistic, ... See [49, 53, 69, 65] for further test statistics.

3. A *level of significance* α is selected which defines how strict the test will be and how much leeway we will accept for the PRN. Common values for α are 0.05 or 0.01.

4. Using the (hopefully) known distribution of the test statistic for the case of H_0 being true, we determine the *critical region* C which is the area in the domain of the test statistic which covers its extreme, unlikely values. The value for α is used to quantify what we mean by "extreme" and "unlikely" in such a way that $P(\text{test statistic} \in C \mid H_0) \leq \alpha$.

5. Now the pseudorandom numbers are generated and are used to to calculate the test statistic. If the value for the test statistic falls into C we say that the PRN have failed the test and we have reason to believe that H_1 is true, otherwise the test is passed and we have no reason to reject H_0.

This is the basic outline of all common empirical tests. We will discuss a few tests and their results later in this thesis. So what are the weak spots in this method of testing pseudorandom numbers ?

- **PRN are *not* experiments.**

  First of all, when testing sequences of numbers generated by a PRNG basic premises of statistical testing are violated.

  The theory behind statistical testing assumes that we actually deal with *random* events. It is thus a circular argument to conclude from the result of

such a test that the numbers in question are "random". Only their statistical properties that are subject to the test, not the basic premise that the concept of random variables is a valid model for that experiment.

It is therefore not correct to speak of "statistical testing" with respect to PRN testing. A more appropriate term is **"numerical testing"** as the test examines only a numerical property of a *fixed* set of numbers.

The only "statistical" part of the test is the calculation what numerical values for the test statistic are considered to be good and which are considered to be bad.

- **The test statistic is arbitrary.**

  The test statistic determines which aspect of the numbers we want to test. Dividing $[0,1[$ into the intervals $\left[\frac{i}{n}, \frac{i+1}{n}\right[$ for $0 \leq i < n$, counting the hits in each interval, and then calculating the χ^2 statistic is a straight-forward test statistic which aims the the overall equidistribution of the pseudorandom numbers in $[0,1[$. The choice of the bins (in this case intervals) seems to be a natural one.

  But what bins should we use to measure the finer aspects of equidistribution ? We could use just a large value for n and keep the intervals, but that would cause a problem with the validity of the χ^2 approximation as the number of hits per bins decreases. An other option is to use something like this: Define bin i as $\{x \in [0,1[: \lfloor x \cdot k \rfloor \equiv i \pmod{n}\}$ for some suitable values for k and n. Then the bins are no longer simple intervals, but sets of intervals that are spread over the unity interval. The value for k determines the width of each component interval.

  Is there a natural choice for k ? We do not think so. But the choice can be important as the result of the test depends on it. Consider for example the set of numbers defined by $\{i/m \mid 0 \leq i < m\}$ which is perfectly equidistributed in$[0,1[$. Fro certain relations between k, n and m, like $k \mid m \cdot n$, then the test will result in extremely bad χ^2 statistics. As an example, consider the case $k = n \cdot m$ where all numbers i/m fall into the same bin. For other values of k, this set of PRN will exhibit no weakness in this test.

  We have seen that even such simple changes to the test statistic, like modifying the width of stripes, can completely change the result of the test. One can imagine that completely different layouts of bins will lead to a great variance in the test results, too. Thus we have to keep in mind that the choice of the test statistic, and thus to some extent the result of the test, is arbitrary.

  One consequence of this fact is that we cannot declare one sequence of PRN to be better than a second one just because we found a test where the

first one rates better, as a slightly modified test might produce exactly the opposite result.

- **Good PRN *have* to fail some tests.**

  What we strive for are numbers who behave in most respects like realisations of $U([0,1[)$-distributed RVs, so it is a natural question how such ideal random numbers will perform in our statistical test. The answer to this is quite simple: If we conduct a test at the significance level α then a sequence of random numbers will fail the test with probability α. If we have set $\alpha = 0.01$ then we can expect a failure about once every hundred tries.

  As PRN should model *all* aspects of real random numbers, they should fail statistical tests at about the same rate.[4]

  > *It is therefore not advisable to outright reject a sequence of PRN based on its failure in single tests.*

- **The variance of the test statistic can be important, too.**

  Classic statistical tests examine if the test statistic does not deviate from its expected value too much. If we are only interested in the expected outcome of a similar simulation problem, such *one-level* statistical tests are all we need in order to be confident about the accuracy of the simulation.

  On the other hand we might be interested in the distribution of the simulation's outcome. For this goal hitting the expected value is not enough, the variance of the result is now important, too. Thus we will demand the same behaviour from the test statistic, too.

  Let us illustrate this principle with an example. We want to test the well known strategy of doubling the ante in a game of roulette. It is supposed to guarantee winning the initial ante and works like this: If we do not win in the first round (and therefore win twice the ante) the ante is doubled for the next round. If this round is won, we get back four times the initial ante while we invested three times the initial ante resulting in a net win of one ante. In case of bad luck we double the ante again hoping for eight times the ante for an investment of seven. As we hope that we will finally win before our capital is drained a net win seems to be certain.

  In order to simulate this we need random numbers to determine whether we will win the current bet. The probabilities are $18/37$ for winning and $19/37$ for losing each round, respectively. It seems to be natural to use the lengths of runs as a test statistic to test our source of PRN for its fitness to simulate a real roulette table. The probability that the maximal run length in 500 tries is greater than 15, is smaller than all usual values for α, so according to

[4] The number of failed tests could be viewed as a test statistic, too. Therefore the sequence of PRN which passes all common level-1 tests will surely fail this meta-test.

the corresponding statistical test we should reject all sequences where such runs do occur.[5]

When we now run the simulation with these prescreened sequences we will *never ever* experience a loss as long as we have enough money for 15 steps of doubling the ante. Thus we should conclude that the strategy works. As we know, this is not true. So what went wrong with our simulation ?

The statistical test considered it equally important whether the sequence in question was "well-behaved" or not, whereas the simulation assigned completely different weights to those cases. Thus the area that the test considered to be insignificant (smaller than α) played a major role in the simulation (more than 1/2).

There are some other cases of simulations where we are not so much interested in the average case, but in the extreme ones. Consider for example all those safety measures in power plants or other machinery where a rare sequence of occurrences might lead to catastrophic results. When simulating these security systems one must not a priori exclude unusual sequences.

Please note that the distinction between level-1 and level-2 tests (tests which test the distribution of the results of a level-1 test) is arbitrary. The test statistic of a level-2 test is just another function of the underlying set of PRN, too.

- **Statistical tests cannot be used to *objectively* rate PRN.**

  One might be tempted to use a set of statistical test to once and for all declare one generator superior to another generator. Intuitively this makes sense, especially when comparing two generators of the same type. A common use for this heuristic is the selection of optimal parameters for the LCG based on its lattice structure [28, 72].

  But is this judgement mathematically justified ? Leeb explains in [58] that such judgement is *not* justified as all possible sequences of PRN of a given finite length pass exactly the same number of statistical tests.

  In order be able to use statistical tests as a criteria for the selection of PRNG the user has to declare which properties he considers important. With this knowledge it is possible to weight the tests and therefore select a suitable generator for this specific application.

- **Statistical tests are simulations with a known result.**

  Both statistical tests and simulations use a set of PRN to perform a more or less elaborate calculation.

$$\text{calculation(PRN)} = \begin{cases} \text{test statistic} \\ \text{simulation result} \end{cases}$$

[5] See [8, p.130] for bounds on these probabilities.

In a statistical test we draw a conclusion from the right side to left: Based on the difference between expected and calculated value we judge the quality of the PRN.

A simulation operates the other way round. Based on the (hopefully) known quality of the PRN we hope that the simulation result is correct.

While generic statistical tests can be used for this reasoning, one can increase their value by designing the tests to closely model the simulation. Thus the tests can target exactly those properties in the PRN which will be significant in their application.

2.3.3 Interpreting Test Results

In order to conclude this chapter on the notion of randomness let us recapitulate what we know about testing a generator, and how we should proceed when we face the task of selecting a generator for a particular simulation problem.

1. Statistical testing *cannot guarantee* that the tested generator will perform equally well in a real world simulation. Only if we are able to determine which aspects of randomness are important for the simulation and specifically test our source of pseudorandom numbers for these aspects, we can mathematically justify our confidence in the validity of the simulation [58].

2. Empirical testing should be supplemented by analytical investigations into the quality of the generator. Such calculations are often essential to guarantee basic properties like period length, as well as to provide means to select suitable parameters to avoid pitfalls like a degenerated lattice in the case of linear generators.

   Furthermore, analytical investigations can yield some insight in the overall structure of a generator's output which can be compared to the properties required in the simulation. The lattice structure of the LCG might be actually useful in Quasi-Monte Carlo integration whereas it can be harmful in geometric problems, e.g. the nearest-pair test [14, 53].

3. Level-one tests target only the expected value of the test statistic. Often this is not enough, making it necessary to test the distribution of the test statistic. Such two-level tests guarantee that the proper irregularities are present in the PRN, too. Whether we should strive for such variance in the generator is up to its final application.

4. Rejecting a generator is rejecting one facet of randomness. It is self-deception to claim that one rejects a generator based on its lack of randomness. One can only state that one does not want that particular aspect of randomness.

Sequences we will intuitively classify as non-random may be important as input for the simulation in order to get correct results.

5. As we have seen before, the results of a test can be extremely sensitive to its parameters. Thus statements like "generator x passes the y-test" have little relevance unless the exact parameters used in the test and in the generator are published, too.

6. The result of a single test is not enough to assess the quality of generator. Only a battery of tests and comparisons to the performance of other generators in the same test suite enable the mathematician to pass judgement on a generator.

Chapter 3

Theoretical Results

In this chapter we will discuss analytically derived properties of the explicit inversive congruential generator (EICG). We will use results obtained for the LCG to serve as a reference as the LCG is the most commonly used generator. Let us start by repeating the definition of the EICG:

Let p be a (large) prime and $a, b, n_0 \in \mathbb{Z}_p$. The explicit inversive congruential generator *(abbreviated as "EICG") with parameters p, a, b, and n_0 defines a sequence $(y_n)_{n\geq 0}$ in $\mathbb{Z}_p$ by*

$$y_n := \overline{a \cdot (n_0 + n) + b} \qquad (n \geq 0) \tag{3.1}$$

and a sequence $(x_n)_{n\geq 0}$ of pseudorandom numbers in $[0, 1[$ by

$$x_n := \frac{y_n}{p} \qquad (n \geq 0).$$

Please remember that we perform all calculations except the final scaling in the finite field $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$. $\mathbb{Z}_p^*$ will denote the non-zero elements of $\mathbb{Z}_p$, that is $\mathbb{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$. The over-line $\overline{a}$ denotes the multiplicative inversion in $\mathbb{Z}_p$ for all non-zero elements $a \in \mathbb{Z}_p^*$. With the special case $\overline{0} = 0$ added, $x \mapsto \overline{x}$ is a bijective function from $\mathbb{Z}_p$ onto $\mathbb{Z}_p$. Furthermore, we have $\overline{\overline{x}} = x$ and $\overline{x} = x^{p-2}$ for all $x \in \mathbb{Z}_p$. The latter identity is due to Fermat's Little Theorem.

Note that from the explicit definition of the sequence $(y_n)_{n\geq 0}$ we can easily

derive a recursive description:

$$\begin{aligned} y_0 &= \overline{a \cdot n_0 + b} \\ y_{n+1} &= \overline{\overline{y_n} + a} \qquad (n \geq 0) \end{aligned} \tag{3.2}$$

In order to achieve maximal period length p, the parameters a,b, and n_0 can be freely chosen from $\mathbb{Z}_p$ as long as p is prime and $a \neq 0$. To see this, consider the function $f(n) := \overline{a \cdot (n_0 + n) + b}$ which is composed of bijective functions in $\mathbb{Z}_p$ and thus is bijective, too. As $n + p = n$ in $\mathbb{Z}_p$ we have $f(n + p) = f(p)$ for all n, thus the sequence $(y_n)_{n=0}^{\infty}$ is purely periodic with period length p.

We will only consider full period generators, that is $a \neq 0$.

We will write $\mathrm{eicg}(p, a, b, n_0)$ to denote the output of a particular instance of the EICG method. Unlike Leeb [58, p. 89] we mean the whole infinite (but periodic) sequence, and not just the first p numbers. This way, no special treatment of wrap-arounds is needed when considering subsequences.

3.1 Relations between different EICG

The choice of parameters is simple for an EICG, but not all choices will lead to completely different pseudorandom numbers. In this section we will examine the relations between EICGs with the same modulus, but different parameters a, b, and n_0.

These results are helpful for the implementation, as one can eliminate an addition modulo p, as well as to the theoretical investigation as they provide a very elegant way to describe sub-streams. We will elaborate on this idea which is due to Niederreiter [70, p. 5] later on.

First of all, let us make a rather trivial observation on the role of the parameter n_0.

Observation 3.1 *Let* $(y_n)_{n\geq 0} = \mathrm{eicg}(p, a, b, 0)$. *Then we can write the sequence* $\mathrm{eicg}(p, a, b, n_0)$ *as* $(y_n)_{n\geq n_0}$. *In other words, the second sequence is first one shifted by* n_0.

Proof: This relation follows from the fact that n and n_0 appear only as their sum $n + n_0$ in the definition of the EICG. ∎

The following observation is taken from Leeb [58, p. 89]; it states that one of the parameters is redundant.

Observation 3.2 *Let p be prime and $a \in \mathbb{Z}_p^*$ be fixed. Then we have for all $b, n_0 \in \mathbb{Z}_p$*

$$\begin{aligned} \mathrm{eicg}(p,a,b,0) &= \mathrm{eicg}(p,a,0,\overline{a}b) && (3.3)\\ \mathrm{eicg}(p,a,0,n_0) &= \mathrm{eicg}(p,a,an_0,0) && (3.4) \end{aligned}$$

and

$$\mathrm{eicg}(p,a,b,n_0) \;=\; \mathrm{eicg}(p,a,0,n_0+\overline{a}b) \;=\; \mathrm{eicg}(p,a,an_0+b,0). \qquad (3.5)$$

Proof: We base the proof on the recursive definition of the EICG. As the recursion does only depend on a, which is constant, it is sufficient to show that the y_0 of these generators are equal. In the first two cases we have

$$\begin{aligned} \overline{a\cdot 0+b} &= y_0 = \overline{a\cdot\overline{a}b+0}\\ \overline{a\cdot n_0+0} &= y_0 = \overline{a\cdot 0+an_0}, \end{aligned}$$

and the third equality translates to

$$y_0 \;=\; \overline{a\cdot n_0+b} \;=\; \overline{a\cdot(n_0+\overline{a}b)} \;=\; \overline{a\cdot 0+(an_0+b)}.$$

■

The third equality can be used to rewrite any EICG as an EICG with $b=0$, but a different value for n_0. Thus the generating formula can always be rewritten as

$$y_n := \overline{a\cdot(n_0'+n)} \qquad (n\geq 0)$$

which saves one addition. The addition $n_0'+n$ can be implemented by simply incrementing the previous value modulo p, thus we need to perform only one increment, one multiplication, one inversion, and one division to generate the next pseudorandom number.

There is an obvious connection between $\mathrm{eicg}(p,a,b,kn_0)$ and $\mathrm{eicg}(p,ka,b,n_0)$, too:[1]

Observation 3.3 *Let p be prime, $a,k \in \mathbb{Z}_p^*$, and $b \in \mathbb{Z}_p$.*
The sequence $\mathrm{eicg}(p,ka,b,n_0)$ *can be obtained by selecting every k-th element from the sequence* $\mathrm{eicg}(p,a,b,kn_0)$.

Proof: The sequence generated by taking every k-th element in the sequence $\mathrm{eicg}(p,a,b,kn_0) = (y_n)_{n\geq 0}$ can be written as $(y_{kn})_{n\geq 0}$. We have

$$y_n = \overline{a\cdot(kn_0+n)+b}$$

[1] This is a generalisation of Lemma 5.3 in [58].

and thus

$$\begin{aligned} y_{kn} &= \overline{a \cdot (kn_0 + nk) + b} \\ &= \overline{ak \cdot (n_0 + n) + b} \\ &= v_n \end{aligned}$$

where $(v_n)_{n \geq 0} = \mathrm{eicg}(p, ka, b, n_0)$. ■

These three observations give us the tools to show that all maximal period EICGs can be derived from the "mother-EICG" $\mathrm{eicg}(p, 1, 0, 0)$ in the following way:

- Start with the sequence $\mathrm{eicg}(p, 1, 0, 0)$.
- Shift the sequence by $a(n_0 + \overline{a}b)$. The result is $\mathrm{eicg}(p, 1, 0, a(n_0 + \overline{a}b))$.
- Take every ath element of that sequence. According to Observation 3.3 we get $\mathrm{eicg}(p, a, 0, n_0 + \overline{a}b)$.
- According to Observation 3.2 this sequence is the same as $\mathrm{eicg}(p, a, b, n_0)$.

Can these insights help us in the theoretical investigation on how samples from an EICG behave under various tests ? Yes, they provide a very convenient and elegant formalism to describe subsequences and various kinds of s-tuples generated from the stream of pseudorandom numbers. With this formalism, the proofs of discrepancy estimates and non-linear properties are very concise.

First of all, we do not need to bother with the parameter n_0 in the theoretical investigation as we can always rewrite the EICG to one with $n_0 = 0$.

Second, any property of a sequence of EICG numbers, which is valid independently of the parameters used, is immediately valid for subsequences consisting of every k-th element. One direct consequence of this is, that once we can prove that pairs of consecutive numbers are uncorrelated for all valid parameters, we can rule out the possibility of long-range correlations at critical distances. See [10, 19] for a discussion of such problems inherent to the LCG.

The third gain, due to Niederreiter [70], is to be able to write almost arbitrary s-tuples formed out of the stream of EICG numbers as **parallel streams**. Such s-tuples as usually used to examine the correlation between successive numbers. For example, the *overlapping serial test* (see page 21) tests the equidistribution of the vectors

$$\mathbf{y}_n = (y_n, y_{n+1}, \ldots, y_{n+s-1}) \in \mathbb{Z}_p^s \tag{3.6}$$

for $n = 0, 1, \ldots, p-1$ in order to test the PRN $(y_i)_{i \geq 0}$ for correlations. If we pick the first coordinate of each vector we get the original sequence. Picking always the second results in the original sequence shifted by one. According to

the above equivalences we can write this shifted sequence as an EICG with the same parameter a, $n_0 = 0$, and a different b. Thus we have

$$\mathbf{y}_n = \left(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(s)}\right) \in \mathbb{Z}_p^s \tag{3.7}$$

where $(y_n^{(i)})_{n\geq 0}$ is the sequence generated by the EICG $\mathrm{eicg}(p, a, a(i-1)+b, 0)$. The obvious generalisation is to allow almost arbitrary EICGs $\mathrm{eicg}(p, a_i, b_i, 0)$ for each coordinate.

In the following, we will prove all statements on the behaviour of s-tuples in terms of these *parallel streams*. For that, we will need to restrict the possible values for the a_i and b_i in order to avoid certain special cases like $a_1 = a_2 \quad \wedge$ $b_1 = b_2$. As we will see later in the various proofs, we need the condition $\overline{a_i}b_i \neq \overline{a_j}b_j$ for all $i \neq j$. Thus we have the following definition:

Definition 3.1 (Parallel Streams) *Let p be prime, $1 \leq s < p$, and $a_1, \ldots, a_s \in \mathbb{Z}_p^*$, $b_1, \ldots, b_s \in \mathbb{Z}_p$ such that $\overline{a_l}b_1, \ldots, \overline{a_s}b_s \in \mathbb{Z}_p$ are distinct. Then we put*

$$y_n^{(i)} = \overline{a_i n + b_i} \qquad \text{for } i = 1, 2, \ldots, s \text{ and } n \geq 0, \tag{3.8}$$

and define a sequence $(\mathbf{y}_n)_{n\geq 0}$ in the s-dimensional affine space over $\mathbb{Z}_p$ by putting

$$\mathbf{y}_n = \left(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(s)}\right) \in \mathbb{Z}_p^s.$$

An interesting special case of parallel streams, which is more general than the overlapping s-tuples considered above, can be obtained as follows. Choose an integer $m \geq 1$ with $\gcd(m, p) = 1$ and integers $0 \leq n_1 < n_2 < \ldots < n_s < p$ and put

$$\mathbf{y}_n = (y_{mn+n_1}, \ldots, y_{mn+n_s}) \in \mathbb{Z}_p^s \text{ for } n \geq 0,$$

where the y_n are as in (3.1). This sequence of points in $\mathbb{Z}_p^s$ can be written in terms of parallel streams according to Definition 3.1 by putting $a_i = am$ and $b_i = an_i + b$ for $1 \leq i \leq s$. It is easy to show that the $\overline{a_i}b_i$ are distinct, thus all results concerning parallel streams are valid for this general method of composing s-tuples, too.

The non-overlapping tuples $\mathbf{y}_n := (y_{sn}, y_{sn+1}, \ldots, y_{sn+s-1})$ are covered by the concept of parallel streams, too. To see this, set $a_i = sa$ and $b_i = a(i-1)+b$ for $i = 1, \ldots, s$.

3.2 Structural Properties

The best known structural property of any pseudorandom number generator is the *lattice structure* of the LCG. Coveyou/MacPherson [9] and Marsaglia [64]

noted first that s-tuples formed from successive LCG-numbers form a lattice in the s-dimensional cube. Figure 3.1 depicts the lattice formed by plotting overlapping 2-tuples for the full period of two "toy" generators. "Production quality" generators exhibit the same structure, you just have to zoom into the image to see the pattern.

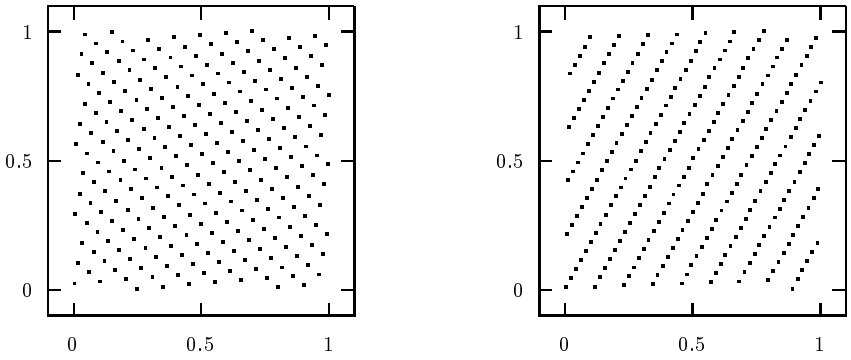


Figure 3.1: Overlapping pairs from lcg(256,69,5,1) and lcg(256,53,1,1)

The shape of the lattice depends very strongly on the parameters a and b of the LCG. Thus various measurements on the *coarseness* of the lattice are used to select suitable parameters a and b. That way, a weakness of the LCG turns into a strength, as one can guarantee a well-behaved lattice for low dimensions as long as the parameters are chosen well enough.

Does the EICG exhibit a similar structure ? Figure 3.2 suggests that the EICG does not possess this linear property, although one can see some other regularities. In fact, one can prove a very stringent non-linearity property for s-tuples taken from an EICG. The theorem describing this is due to Niederreiter [70].

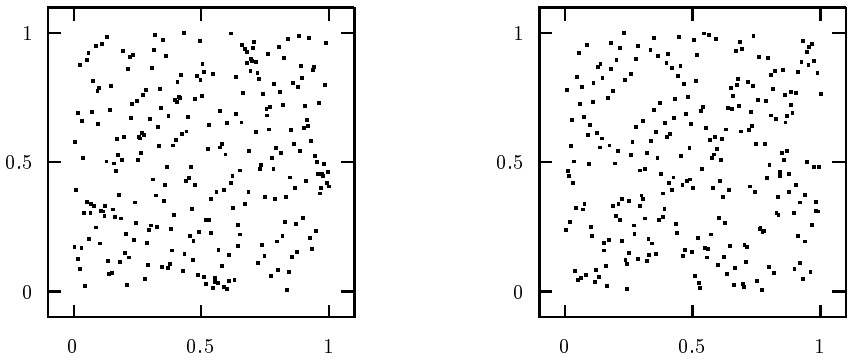


Figure 3.2: Overlapping pairs from eicg(257,6,1,0) and eicg(257,30,1,1,0)

Theorem 3.1 *Let* $\mathbf{y}_n = \left(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(s)}\right)$ *as in Definition 3.1, then every hyperplane in* $\mathbb{Z}_p^s$ *contains at most* s *of the points* $\mathbf{y}_n$, $n = 0, 1, \ldots, p-1$, *with* $y_n^{(1)} \cdots y_n^{(s)} \neq 0$. *If the hyperplane passes through the origin of* $\mathbb{Z}_p^s$, *then it contains at most* $s-1$ *of these points* $\mathbf{y}_n$.

Proof: All calculations in this proof are performed in the finite field $\mathbb{Z}_p$. Furthermore, remember that according to Definition 3.1, the $\overline{a_i}b_i$ are distinct.

A hyperplane $H_{\mathbf{c},c_0}$ in Z_p^s is uniquely defined by a vector $\mathbf{c} = (c_1, \ldots, c_s) \in \mathbb{Z}_p^s \setminus \{\mathbf{0}\}$ and a scalar $c_0 \in \mathbb{Z}_p$ as $H_{\mathbf{c},c_0} = \{\mathbf{x} \in Z_p^s \mid \mathbf{x} \cdot \mathbf{c} = c_0\}$. We restrict our search for points on a hyperplane $n \in W := \mathbb{Z}_p \setminus \{-\overline{a_1}b_1, \ldots, -\overline{a_s}b_s\}$. Thus for $n \in W$ we have according to (3.8) $y_n^{(1)}, \ldots, y_n^{(s)} \neq 0$, therefore we can rewrite the hyperplane equation for $\mathbf{y}_n$ as

$$\sum_{j=1}^{s} \frac{c_j}{a_j n + b_j} = c_0.$$

By clearing denominators, we see that n is a root of the polynomial

$$h(x) = c_0 \prod_{i=1}^{s} (a_i x + b_i) - \sum_{j=1}^{s} c_j \prod_{\substack{i=1 \\ i \neq j}}^{s} (a_i x + b_i).$$

If $c_0 \neq 0$, then h is a nonzero[2] polynomial of degree s over $\mathbb{Z}_p$. Since such a polynomial has at most s roots in $\mathbb{Z}_p$, the hyperplane $H_{\mathbf{c},c_0}$ contains at most s of the $\mathbf{y}_n$ with $n \in \{0, 1, \ldots, p-1\} \setminus \{-\overline{a_1}b_1, \ldots, -\overline{a_s}b_s\}$.

If $c_0 = 0$, that is $\mathbf{0} \in H_{\mathbf{c},c_0}$, we get

$$h(x) = \sum_{j=1}^{s} c_j \prod_{\substack{i=1 \\ i \neq j}}^{s} (a_i x + b_i),$$

whose degree is at most $s-1$. It remains to show that h is not the zero polynomial. As $\mathbf{c}$ is not the zero vector, one of its coordinates is nonzero. For $c_k \neq 0$ we have

$$\begin{aligned} h(-\overline{a_k}b_k) &= c_k \prod_{\substack{i=1 \\ i \neq k}}^{s} (-a_i \overline{a_k} b_k + b_i) \\ &= c_k \prod_{\substack{i=1 \\ i \neq k}}^{s} a_i (\overline{a_i} b_i - \overline{a_k} b_k) \\ &\neq 0, \end{aligned}$$

because c_k is the chosen nonzero coordinate, and the a_i as well as the $(\overline{a_i}b_i - \overline{a_k}b_k)$ are nonzero according to the conditions of the theorem. As we have found $h(x) \neq 0$ for some x, h cannot be the zero polynomial. ∎

[2]Remember: $a_i \neq 0$ for $i = 1, \ldots, s$.

This theorem proves that s-tuples taken from an EICG do not form any linear structure such as a lattice. But that does not mean that *no* other kind of pattern emerges in plots of pairs of consecutive[3] numbers. For example, consider Figure 3.3, where one can see a hyperbola-like structure in the upper left and lower right corner. Eichenauer-Herrmann and Wegenkittl are currently preparing a paper discussing these properties of the EICG.

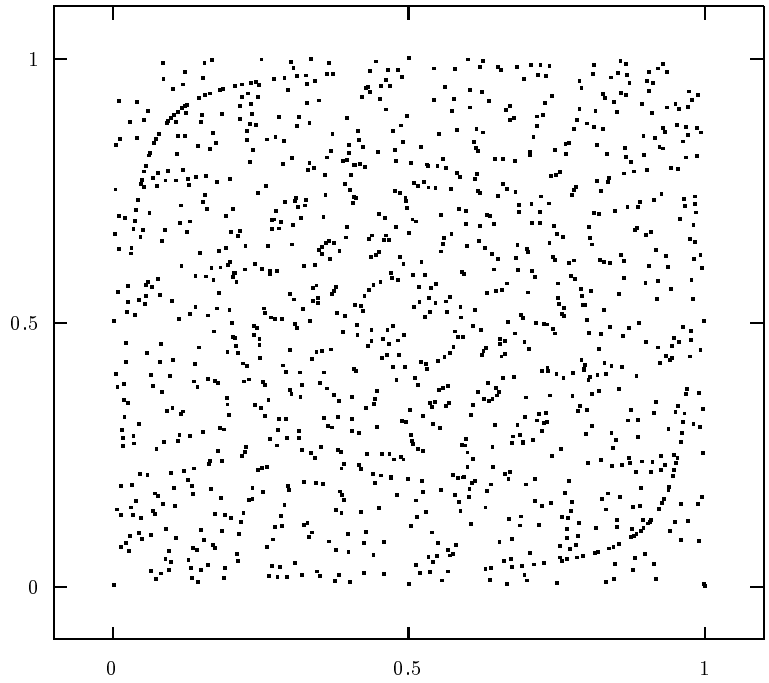


Figure 3.3: Overlapping pairs from eicg(1163,1,0,0)

All the plots so far contained all the overlapping pairs available from the *full period* of the generator. This way, the underlying structure of the generator is perfectly visible. But usually one does not utilize the full period of any generator; a common rule of thumb is to use not more than the square root of the period. Thus the LCG will never be able to build up the full lattice and the EICG will contain only a few points on hyperbola. Figure 3.4 depicts the lattice of lcg(65536,325,1,1), the first image shows the full lattice in a zoomed view, the second one contains only 256 points, which corresponds to the square root of all possible points.

[3]Note that according to Observation 3.3 any step corresponds to a single step for a different EICG, thus we can restrict our search for patterns to pairs of consecutive numbers.

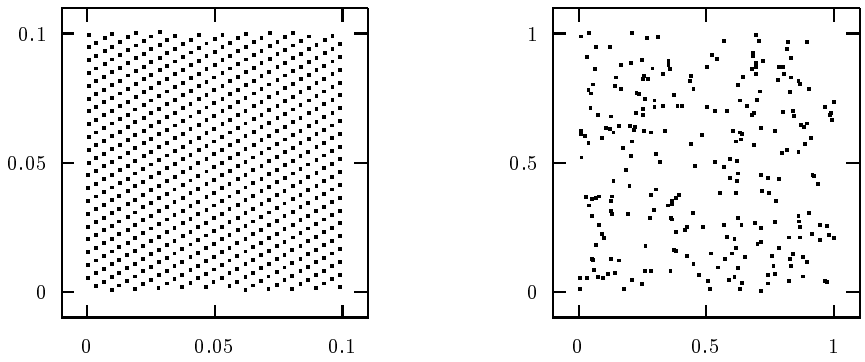


Figure 3.4: Overlapping pairs from lcg(65536,325,1,1)

These figures clearly demonstrate that any regularities a generator develops over the full period are not necessarily present when only a fraction of the available numbers is used. The recommendation never to exhaust the full period can be further justified by the following argument: The PRNG is supposed to simulates drawing numbers from an urn *with* putting the numbers back into the urn, but in fact the typical PRNG empties the imaginary urn before it puts *all* numbers back when the period is exhausted. The difference between "drawing with replacement" and "drawing without replacement" is small as long as only a fraction of all numbers are drawn from the urn.

3.3 Correlation Analysis

The *discrepancy* is a widely used and well studied measure for the equidistribution of a set of points. In this section we will try to give a motivated definition, some theoretical background, and summarize all published results concerning the EICG.

3.3.1 Background

There are at least three approaches to the notion of discrepancy, one stems from statistics, one from geometric reasoning, and one from numerical integration. We will use the latter. An extensive introduction to discrepancy can be found in Niederreiter [69, Chapter 2].

We will use the following setting: The closed s-dimensional unit cube $\bar{I}^s = [0,1]^s$ will be the integration domain in which we will try to integrate the function f by using the quasi-Monte Carlo integration

$$\int_{\bar{I}^s} f(\mathbf{u})d\mathbf{u} \approx \frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n) \tag{3.9}$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \bar{I}^s$. Ideally, we hope that the approximation converges to the integral as the number of points increases. If this is the case for a reasonable class of functions, say, for all continuous f on $\bar{I}^s$, then we call the (infinite) sequence $(\mathbf{x}_1, \mathbf{x}_2, \ldots)$ *uniformly distributed* in $\bar{I}^s$. One can show that this definition of "uniformly distributed" is quite independent of the class of functions; using the Riemann-integrable functions yields the same test as using the characteristic functions of a *very simple* set of intervals.

Whereas the limit of the integration error can be used as a *qualitative* measure for the distribution properties of an (infinite) sequence of points, one can use the integration error in the finite case as a *quantitative* measurement of the equidistribution of the finite sequence $(\mathbf{x}_n)_{n=1}^N$.

In order to get a workable measurement, we have to state which family of functions f we consider for the integration, and how we condense all the integration-errors for each function from the family into one single number.

The general concept of *discrepancy* uses the set of characteristic functions of axis-parallel cubes in $I^s := [0,1[^s$ as the functions to integrate and the supremum as the condensing function. Formally, we can write this in the following way:

If $\Omega = (\mathbf{x}_n)_{n=1}^N$ is a finite sequence in I^s, and B an arbitrary subset of I^s, then we can express the quasi-Monte Carlo integration of the characteristic function[4] c_B in terms of the number of $\mathbf{x}_i$ in B,

$$A(B;\Omega) := \#\{n \in \{1,\ldots,N\} \mid \mathbf{x}_n \in B\},$$

as

$$\int_{\bar{I}^s} c_B(\mathbf{u})d\mathbf{u} \approx \frac{1}{N}\sum_{n=1}^{N} c_B(\mathbf{x}_n) = \frac{1}{N}A(B;\Omega).$$

Based on this, the error when integrating c_B can be written as $\left|\frac{A(B,\Omega)}{N} - \lambda_s(B)\right|$, where $\lambda_s(B)$ is the s-dimensional volume[5] of B. Thus we can write the general notion of the discrepancy of a finite sequence Ω of points in I^s for an arbitrary[6] family $\mathcal{B}$ of sets as

$$D_N(\mathcal{B};\Omega) = \sup_{B\in\mathcal{B}}\left|\frac{A(B,\Omega)}{N} - \lambda_s(B)\right| \tag{3.10}$$

From this general definition we can derive the definition of the two most important incarnations of discrepancy as follows:

Definition 3.2 *The **star discrepancy** $D_N^*(\Omega) = D_N^*(\mathbf{x}_1, \ldots \mathbf{x}_N)$ of the finite sequence Ω is defined by $D_N^*(\Omega) := D_N(\mathcal{J}^*;\Omega)$, where $\mathcal{J}^*$ is the family of all subintervals of I^s of the form $\prod_{i=1}^s [0, u_i[$.*

[4] $c_B(\mathbf{x})$ is defined as 1 for $\mathbf{x} \in B$ and 0 for $\mathbf{x} \notin B$.

[5] To be exact, $\lambda_s(B)$ denotes the s-dimensional Lebesgue-measure of B.

[6] $\mathcal{B}$ should be a non-empty family of Lebesgue-measurable subsets of I^s.

Definition 3.3 *The **(extreme) discrepancy** $D_N(\Omega) = D_N(\mathbf{x}_1, \ldots \mathbf{x}_N)$ of the finite sequence Ω is defined by $D_N(\Omega) := D_N(\mathcal{J}; \Omega)$, where $\mathcal{J}$ is the family of all subintervals of I^s of the form $\prod_{i=1}^{s} [u_i, v_i[$.*

While D_N and D_N^* are the classical figures for measuring the equidistribution, they are far from being the only ones. Interesting variations of the basic idea are the *isotropic discrepancy*, which uses the family of convex sets instead of axis-parallel cubes, or the *L^2-discrepancy*, which uses the 2-norm instead of the supremum. Especially the L^2-discrepancy has received a lot of attention recently as it is suitable for empirical testing [37] and has a number of interesting theoretical properties [79, 61]. Another measurement worth mentioning is the weighted spectral test [39, 45, 43, 40, 46].

Let us quickly state a few general results concerning discrepancy. They will help us to interpret the main results of this chapter. We once again refer to Niederreiter [69, p. 14ff, p. 166ff] for proofs and further references.

Proposition 3.1 *For all finite sequences Ω consisting of N points in $\bar{I}^s$ we have*

$$D_N^*(\Omega) \leq D_N(\Omega) \leq 2^s D_N^*(\Omega)$$

and

$$0 \leq D_N^*(\Omega) \leq D_N(\Omega) \leq 1.$$

In dimension one, that is $s = 1$, it is possible to express the discrepancy as a relatively simple formula operating on the *ordered list* of points.

Proposition 3.2 *If $0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq 1$, then*

$$D_N^*(x_1, \ldots, x_N) = \frac{1}{2N} + \max_{1 \leq n \leq N} \left| x_n - \frac{2n-1}{2N} \right|$$

and

$$D_N(x_1, \ldots, x_N) = \frac{1}{N} + \max_{1 \leq n \leq N} \left(\frac{n}{N} - x_n \right) - \min_{1 \leq n \leq N} \left(\frac{n}{N} - x_n \right).$$

From these formulae, as well as the well known fact that sorting is of complexity $\mathcal{O}(N \log N)$, it is easy to see that one can calculate the discrepancy in the one-dimensional case in $\mathcal{O}(N \log N) + \mathcal{O}(N)$ steps. Using a memory versus speed tradeoff [33] it is possible to get the complexity down to $\mathcal{O}(N)$. In higher dimensions s calculating the discrepancy is of complexity $\mathcal{O}(N^s)$, making any reasonable empirical testing computationally infeasible. Probabilistic algorithms [88] can be employed to calculate tight upper bounds for a given Ω.

What do we know about the behaviour of D_N with increasing N ? If the sequence of point is indeed uniformly distributed in $\bar{I}^s$, then we know that

$\lim_{N\to\infty} D_N = 0$. For a sequence of uniformly distributed *random* points we know that $\lim_{N\to\infty} D_N = 0$ with probability one. But in order to use the discrepancy as a figure of merit for *finite* sequences, we need to know exactly *how* D_N converges for random sequences. Luckily, the following result (due to Kiefer [48]) provides us with the benchmark according to which we can judge the discrepancy bounds derived for PRN.

Proposition 3.3 (Law of the iterated logarithm) *Let $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$ be a sequences of uniformly distributed random points in $\bar{I}^s$, then we have*

$$\overline{\lim}_{N\to\infty} \frac{(2N)^{1/2} D_N^*(\mathbf{z}_1, \ldots, \mathbf{z}_N)}{(\log\log N)^{1/2}} = 1 \quad \lambda^\infty - a.e.,$$

where λ^∞ is the Lebesgue measure on the space of all infinite sequences in $\bar{I}^s$.

The discrepancy is per definition an upper bound for the quasi-Monte Carlo integration error for a very limited class of functions, namely the characteristic functions of axis-parallel cubes. A classic result by Hlawka uses the discrepancy to derive an error-bound for a large class of functions.

Proposition 3.4 (Koksma-Hlawka inequality [69]) *If f has bounded variation $V(f)$ on $\bar{I}^s$ in the sense of Hardy and Krause, then for any $\mathbf{x}_1, \ldots, \mathbf{x}_N \in I^s$ we have*

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_N) - \int_{\bar{I}^s} f(\mathbf{u}) d\mathbf{u} \right| \leq V(f)\, D_N^*(\mathbf{x}_1, \ldots, \mathbf{x}_N).$$

Why is this inequality so important ? For the Monte Carlo numerical integration, which is based on "random numbers", one cannot derive an a-priori[7] error bound on the integration error. It is only possible to state a probabilistic error bound, a shortcoming that is often not acceptable. The inequality of Koksma-Hlawka on the other hand, is a *hard* bound on the integration error. Thus in order to get such a bound for the Monte Carlo method, one has to calculate the discrepancy for the numbers used, which is not feasible in practice. The way out is to use a set of numbers for which bounds on the discrepancy are known in advance, such as (t, m, s)-nets or PRNGs for which such bounds are available.

On the other hand, if we want our PRNG to model a $U[0, 1[$ distributed random variable as closely as possible, the law of the iterated logarithm provides us with the correct order of magnitude for the discrepancy. One can argue that any results concerning the discrepancy of a particular generator which shows a rate of growth close to $\mathcal{O}(N^{-1/2} \log\log N)$ is a sign for the right amount of "randomness" in the generator. Empirical evidence seems to support this argument. In any case, the discrepancy is certainly the most widely used figure of merit in theoretical analysis of pseudorandom number generation algorithms.

[7] A-priori in the sense of "before the random numbers are actually drawn".

3.3.2 Auxiliary Results

In order to prove discrepancy bounds, we need a variety of auxiliary results. Unfortunately it is not possible to include the proofs for these lemmata without exceeding the scope of this thesis, as well as alienating the target audience. Thus we will only list the results and give references to the proofs.

The basic approach to derive discrepancy bounds for PRNG is due to Niederreiter [69], who established a link between the discrepancy of a finite sequence of points with rational coordinates and certain exponential sums. As most common PRNG use integer arithmetic combined with a final scaling operation, this approach is perfectly suited for the study of the output of such generators.

Niederreiter's proof [69, §3.2] is elementary, although rather tricky. Hellekalek [38] gave a proof based on dyadic harmonic analysis. A detailed proof can be found in Weingartner [86].

In correspondence with the literature [70, 69], we will use the following definitions: For a prime $p \geq 5$ let $C_s^*(p)$ be the set of all nonzero $\mathbf{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$ with $|h_i| < p/2$ for $1 \leq i \leq s$. For such $\mathbf{h}$, put $r(\mathbf{h}, p) = \prod_{i=1}^{s} r(h_i, p)$ with $r(h, p) = p \sin(\pi |h|/p)$ for h nonzero and $r(0, p) = 1$. Furthermore, set $\chi(n) = e^{2\pi\sqrt{-1}\, n/p}$ for $n \in Z_p$, and let $\mathbf{u} \cdot \mathbf{v}$ denote the standard inner product of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^s$.

Although some of the Lemmata below do not need this restriction, we will consider only the case of prime moduli here.

Lemma 3.1 *For a prime $p \geq 2$ and $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{N-1} \in \mathbb{Z}_p^s$, let Ω be the finite sequence $(\mathbf{x}_n = \mathbf{y}_n/p)_{n=0}^{N-1}$. Then*

$$D_N(\Omega) \leq 1 - (1 - 1/p)^s + \sum_{\mathbf{h} \in C_s^*(p)} \frac{1}{r(\mathbf{h}, p)} \left| \frac{1}{p} \sum_{n=0}^{N-1} \chi(\mathbf{h} \cdot \mathbf{y}_n) \right|.$$

The following Lemma is due to Niederreiter [70, Lemma 2] which improves [69, Corollary 3.11].

Lemma 3.2 *Let $p \geq 5$ be a prime and let $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{N-1} \in \mathbb{Z}_p^s$. Suppose the real number B is such that*

$$\left| \sum_{n=0}^{N-1} \chi(\mathbf{h} \cdot \mathbf{y}_n) \right| \leq B \text{ for all } \mathbf{h} \in C_s^*(p).$$

Then the discrepancy D_N of the finite sequence $(\mathbf{x}_n = \mathbf{y}_n/p)_{n=0}^{N-1}$ satisfies

$$D_N \leq 1 - (1 - 1/p)^s + \frac{B}{N} \left(\frac{4}{\pi^2} \log p + 1.38 + \frac{0.64}{p} \right)^s.$$

In order to derive the bound B for the EICG we need the following two Lemmata, the first one is due to Cochrane [7], the second is a variant of the Bomberi-Weil bound for exponential sums (see Moreno and Moreno [67, Theorem 2]):

Lemma 3.3 *For any prime $p \geq 5$ and any integer N we have*

$$\sum_{u=1}^{p-1} \left| \frac{\sin(\pi u N/p)}{\sin(\pi u/p)} \right| < \frac{4}{\pi^2} p \log p + (0.38)p + 0.64.$$

Lemma 3.4 *Let Q/R be a rational function over $\mathbb{Z}_p$ which is not of the form $A^p - A$ with $A \in \overline{\mathbb{Z}}_p(x)$. Let s be the number of distinct roots of the polynomial R in $\overline{\mathbb{Z}}_p$. Then we have*

$$\left| \sum_{\substack{b \in \mathbb{Z}_p \\ R(n) \neq 0}} \chi\left(\frac{Q(n)}{R(n)}\right) \right| \leq (\max(\deg(Q), \deg(R)) + s^* - 2)p^{1/2} + \delta,$$

where $s^* = s$ and $\delta = 1$ if $\deg(Q) \leq \deg(R)$, and $s^* = s + 1$ and $\delta = 0$ otherwise.

The Koksma-Hlawka inequality can be used to derive a lower bound on the discrepancy of a finite sequence of points. All that is needed is a function with a known integral and bounded variation. In light of the previous lemmata it seems natural to use a function for which the Monte Carlo integration can be expressed in terms of exponential sums. The following result is due to Niederreiter [69, Cor. 3.17].

Lemma 3.5 *For a prime $p \geq 2$ and $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{N-1} \in \mathbb{Z}_p^s$, let Ω be the finite sequence $(\mathbf{x}_n = \mathbf{y}_n/p)_{n=0}^{N-1}$. Then, for any nonzero $\mathbf{h} \in \mathbb{Z}^s$, we have*

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \chi(\mathbf{h} \cdot \mathbf{y}_n) \right| \leq \frac{2}{\pi}((\pi + 1)^m - 1) r(\mathbf{h}) D_N(\Omega),$$

where m is the number of nonzero coordinates of $\mathbf{h}$.

Unfortunately, we cannot prove a lower bound on the generic sum $\sum \chi(\mathbf{h} \cdot \mathbf{y}_n)$ for finite sequences of points generated by an EICG. But we are able to prove such bounds for a slightly different exponential sum $E_N(\chi; \mathbf{d}, \mathbf{e})$, which we can link to sequences generated by an EICG, and which is a special case of the sum in Lemma 3.5. This approach is due to Eichenauer-Herrmann and Niederreiter [22]. All lemmata and theorems concerning lower bounds are taken from this paper.

For $\mathbf{d} = (d_1, \ldots, d_s) \in \mathbb{Z}_p^s$ and $\mathbf{e} = (e_1, \ldots, e_s) \in \mathbb{Z}_p^s$ we define

$$E_N(\chi; \mathbf{d}, \mathbf{e}) := \sum_{n=0}^{N-1} \chi\left(\sum_{j=1}^{s} d_j \overline{n + e_j} \right) \qquad \text{for } 1 \leq N \leq p, \tag{3.11}$$

and put $E(\chi; \mathbf{d}, \mathbf{e}) := E_p(\chi; \mathbf{d}, \mathbf{e})$.

Lemma 3.6 *If* $\mathbf{d} \neq \mathbf{0}$ *and* $e_1, \ldots, e_s$ *distinct, then*

$$|E(\chi; \mathbf{d}, \mathbf{e})| \leq (2s-2)p^{1/2} + s + 1$$

and

$$|E_N(\chi; \mathbf{d}, \mathbf{e})| \leq 2sp^{1/2}\left(\frac{4}{\pi^2}\log p + 0.38 + \frac{0.64}{p}\right) + \frac{N}{p}\left((2s-2)p^{1/2}+1\right) + s.$$

See [22] (Theorem 1 and Corollary 1) for the proofs, which are similar to the proofs of Theorems 3.2 and 3.3. Besides these upper bounds, we know the average value of the $E_N(\chi; \mathbf{d}, \mathbf{e})$, according to the next Lemma.

Lemma 3.7 *Let* $1 \leq N \leq p$ *and* $1 \leq k \leq s$, $\mathbf{e}, \mathbf{d} \in \mathbb{Z}_p^s$ *with fixed* $d_{k+1}, \ldots, d_s$. *Then we have*

$$\sum_{d_1,\ldots,d_k \in \mathbb{Z}_p} |E_N(\chi; \mathbf{d}, \mathbf{e})|^2 = Np^2.$$

Proof: We set $d_{j,m,n} := d_j(\overline{n+e_j} - \overline{m+e_j})$. With $\mathbf{e} = (e_1, \ldots, e_s)$ we get

$$\begin{aligned}
&\sum_{d_1,\ldots,d_k \in \mathbb{Z}_p} |E_N(\chi; \mathbf{d}, \mathbf{e})|^2 = \\
&= \sum_{d_1,\ldots,d_k \in \mathbb{Z}_p} \left| \sum_{n=0}^{N-1} \chi\left(\sum_{j=1}^{s} d_j \overline{n+e_j}\right)\right|^2 \\
&= \sum_{d_1,\ldots,d_k \in \mathbb{Z}_p} \sum_{n,m=0}^{N-1} \chi\left(\sum_{j=1}^{s} d_{j,m,n}\right) \\
&= \sum_{n,m=0}^{N-1} \sum_{d_1,\ldots,d_k \in \mathbb{Z}_p} \chi\left(\sum_{j=1}^{s} d_{j,m,n}\right) \\
&= \sum_{n,m=0}^{N-1} \sum_{d_1,\ldots,d_k \in \mathbb{Z}_p} \prod_{j=1}^{s} \chi(d_{j,m,n}) \\
&= \sum_{n,m=0}^{N-1} \left(\prod_{j=k+1}^{s} \chi(d_{j,m,n})\right) \sum_{d_1,\ldots,d_k \in \mathbb{Z}_p} \prod_{j=1}^{k} \chi(d_{j,m,n}) \\
&= \sum_{n,m=0}^{N-1} \chi\left(\sum_{j=k+1}^{s} \chi(d_{j,m,n})\right) \sum_{d_1,\ldots,d_{k-1} \in \mathbb{Z}_p} \sum_{d_k \in \mathbb{Z}_p} \prod_{j=1}^{k} \chi(d_{j,m,n}) \\
&= \sum_{n,m=0}^{N-1} \chi\left(\sum_{j=k+1}^{s} \chi(d_{j,m,n})\right) \sum_{\substack{d_1,\ldots,d_{k-1} \\ \in \mathbb{Z}_p}} \prod_{j=1}^{k-1} \chi(d_{j,m,n}) \sum_{d \in \mathbb{Z}_p} \chi(d(\overline{n+e_k} - \overline{m+e_k})) \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,m=0}^{N-1} \chi\left(\sum_{j=k+1}^{s} \chi(d_{j,m,n})\right) \prod_{j=1}^{k} \left(\sum_{d\in\mathbb{Z}_p} \chi(d(\overline{n+e_j} - \overline{m+e_j}))\right) \\
&= \sum_{\substack{n,m=0 \\ n=m}}^{N-1} \chi(0) \prod_{j=1}^{k} \left(\sum_{d\in\mathbb{Z}_p} \chi(0)\right) \\
&= Np^k,
\end{aligned}
$$

because we have $\sum_{d\in\mathbb{Z}_p} \chi(d\cdot k) = 0$ for $k \in Z_p, k \neq 0$ and p otherwise. ■

Applying Lemma 3.5 on a finite sequence of points generated by parallel streams of EICG according to Definition 3.1, and $\mathbf{h} = (1, 1, 0, \ldots, 0) \in \mathbb{Z}^s$ we get for $s \geq 2$

$$
\begin{aligned}
D_N^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N) &\geq \frac{1}{2(\pi+2)N} \left| \sum_{n=0}^{N-1} \chi(y_n^{(1)} + y_n^{(2)}) \right| \\
&= \frac{1}{2(\pi+2)N} \left| \sum_{n=0}^{N-1} \chi(\overline{a_1 n + b_1} + \overline{a_2 n + b_2}) \right| \\
&= \frac{1}{2(\pi+2)N} |E_N(\chi; \mathbf{d}, \mathbf{e})| \qquad (3.12)
\end{aligned}
$$

with $\mathbf{d} = (\overline{a_1}, \overline{a_2}) \in \mathbb{Z}_p^2$ and $\mathbf{e} = (b_1\overline{a_1}, b_2\overline{a_2}) \in \mathbb{Z}_p^2$. Similarly, for $\mathbf{h} = (1, 0, \ldots, 0) \in \mathbb{Z}^s$ and $s \geq 1$ we have

$$
D_N^{(s)}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \geq \frac{1}{2N} |E_N(\chi; \overline{a_1}, b_1\overline{a_1})|. \qquad (3.13)
$$

3.3.3 Bounds

We now have the tools necessary to derive upper and lower bounds on the discrepancy of finite sequences of points generated by parallel streams of EICG numbers. But as a reference, let us first look at the result available for the LCG, which will once again serve as a reference.

According to [69, Theorem 7.4] we have for the multiplicative linear congruential generator the following statement: For $s \geq 2$ and for an average multiplier a, the discrepancy of the finite sequence of s-dimensional points consisting of all $M-1$ overlapping tuples from $\mathrm{lcg}(M, a, 0, 1)$ obeys

$$
D^{(s)} = \mathcal{O}\left(M^{-1}(\log M)^s \log\log(M+1)\right),
$$

where the implied constant depends only on s.

Upper Bounds

We first turn our attention to upper bounds; we will prove both bounds for the full, as well as or parts of the period. These bounds are relevant in two ways:

1. A small discrepancy in the s-dimensional case guarantees the uncorrelatedness of these s-tuples. Basically, the EICG passes a generalized serial test, as the parallel streams encompass more than just tuples formed from consecutive numbers.

2. For s-dimensional quasi-Monte Carlo integration, a bound on the discrepancy equates to a bound on the integration error.

Please note that the following theorems do *not* depend on the choice of any parameters; they are valid for every single full period EICG. This is in sharp contrast to the bounds known for the LCG, which is only deals with the *average* over all multipliers, and thus tells us nothing about a particular generator. Furthermore, in the case of the LCG no bounds are known for parts of the period in dimensions $s \geq 2$.

As one can guess from the lemmata listed above, the proofs involve exponential sums which need to be rearranged in a way to be able to use Lemma 3.4.

The following two theorems are due to Niederreiter [70].

Theorem 3.2 *For $p \geq 5$, prime, and $2 \leq s < p$ set $\mathbf{x}_n = \mathbf{y}_n/p$ based on the $\mathbf{y}_n$ of Definition 3.1. Then we have for the discrepancy of the finite sequence $(\mathbf{x}_n)_{n=0}^{p-1}$ the following upper bound:*

$$D_p^{(s)} \leq 1 - (1-1/p)^s + \left(\frac{2s-2}{p^{1/2}} + \frac{s+1}{p}\right)\left(\frac{4}{\pi^2}\log p + 1.38 + \frac{0.64}{p}\right)^s$$

Proof: For $\mathbf{h} = (h_1, \ldots, h_s) \in C_s^*(p)$ put

$$S(\mathbf{h}) = \sum_{n \in \mathbb{Z}_p} \chi(\mathbf{h} \cdot \mathbf{y}_n) = \sum_{n \in \mathbb{Z}_p} \chi\left(\sum_{i=1}^{s} h_i y_n^{(i)}\right).$$

We now restrict the sum to those terms where $y_n^{(i)} \neq 0$ by using the same set W as in the proof of Theorem 3.1 as the summation domain. By noting that $\mathrm{card}(\mathbb{Z}_p \setminus W) = s$ and using the triangle inequation we get

$$|S(\mathbf{h})| \leq s + \left|\sum_{n \in W} \chi\left(\sum_{i=1}^{s} h_i \overline{a_i n + b_i}\right)\right| = s + \left|\sum_{\substack{n \in \mathbb{Z}_p \\ R(n) \neq 0}} \chi\left(\frac{Q(n)}{R(n)}\right)\right|,$$

where Q/R is the rational function over $\mathbb{Z}_p$ given by

$$\frac{Q(x)}{R(x)} = \sum_{i=1}^{s} \frac{h_i}{a_i x + b_i} \quad \text{with } R(x) = \prod_{i=1}^{s}(a_i x + b_i). \tag{3.14}$$

As all a_i are nonzero, we have $\deg(R) = s < p$. Furthermore, as at least one of the h_i is nonzero, the uniqueness of the partial fraction decomposition for rational functions implies that $Q \neq 0$ and $\deg(Q) < s = \deg(R)$. In order to apply Lemma 3.4, we have to show that Q/R is not of the form $A^p - A$ with $A \in \overline{\mathbb{Z}}_p(x)$, where $\overline{\mathbb{Z}}_p$ denotes the algebraic closure of the field $\mathbb{Z}_p$, and $\overline{\mathbb{Z}}_p(x)$ denotes the field of rational functions over $\overline{\mathbb{Z}}_p$. If this were the case, we would have

$$\frac{Q}{R} = \left(\frac{K}{L}\right)^p - \frac{K}{L}$$

with polynomials K, L over $\overline{\mathbb{Z}}_p$ and $\gcd(K, L) = 1$, and thus

$$L^p Q = (K^{p-1} - L^{p-1})KR. \tag{3.15}$$

Since we have demanded $\gcd(K, L) = 1$, L cannot divide K or $(K^{p-1} - L^{p-1})$, thus L^p must divide R. As $\deg(R) = s < p$, that can only be the case if L is a nonzero constant polynomial which implies $\deg(L^p) = 1$. Comparing the degrees in (3.15) yields $\deg(Q) \geq \deg(R)$, which contradicts the degrees derived from (3.14). Thus we can apply Lemma 3.4, and this leads to

$$S(\mathbf{h}) \leq (2s-2)p^{1/2} + s + 1 \quad \text{for all } \mathbf{h} \in C_s^*(p). \tag{3.16}$$

The rest follows from Lemma 3.2. ∎

Theorem 3.3 *For $p \geq 5$, prime, and $2 \leq s < p$ let the finite sequence $(\mathbf{x}_n = \mathbf{y}_n/p)_{n=0}^{p-1}$ be like in Theorem 3.2. Then we have for the discrepancy of the first N points the following upper bound:*

$$\begin{aligned} D_N^{(s)} < \quad & 1 - (1 - 1/p)^s \\ & + \left(\frac{2s-2}{p^{1/2}} + \frac{s+1}{p} + \frac{s}{N}(2p^{1/2}+1)\left(\frac{4}{\pi^2}\log p + 0.38 + \frac{0.64}{p}\right)\right) \cdot \\ & \cdot \left(\frac{4}{\pi^2}\log p + 1.38 + \frac{0.64}{p}\right)^s \end{aligned}$$

Proof: Just as in the proof of Theorem 3.2 we need to derive a bound for an exponential sum to be able to apply Lemma 3.2. This time the summation domain is not $\mathbb{Z}_p$, thus we need to rewrite the sum in a rather tricky way.

For $\mathbf{h} = (h_1, \ldots, h_s) \in C_s^*(p)$ put

$$S_N(\mathbf{h}) = \sum_{n=0}^{N-1} \chi(\mathbf{h} \cdot \mathbf{y}_n) = \sum_{n=0}^{N-1} \chi(r_n)$$

with $r_n = \sum_{i=1}^{s} h_i y_n^{(i)}$ for $n \geq 0$. We can now rewrite $S_N(\mathbf{h})$ using the fact that $\sum_{u \in \mathbb{Z}_p} \chi(uk)$ evaluates to 0 for $k \neq 0$ and to p otherwise.

$$\begin{aligned}
S_N(\mathbf{h}) &= \sum_{n=0}^{N-1} \chi(r_n) \\
&= \sum_{n=0}^{p-1} \chi(r_n) 1_{\{0,\ldots,N-1\}}(n) \\
&= \sum_{n=0}^{p-1} \chi(r_n) \sum_{t=0}^{N-1} \delta_{t,n} \\
&= \sum_{n=0}^{p-1} \chi(r_n) \sum_{t=0}^{N-1} \frac{1}{p} \sum_{u=0}^{p-1} \chi(u(n-t)) \\
&= \frac{1}{p} \sum_{u=0}^{p-1} \sum_{t=0}^{N-1} \sum_{n=0}^{p-1} \chi(r_n)\chi(-ut)\chi(un) \\
&= \frac{1}{p} \sum_{u=0}^{p-1} \left(\sum_{t=0}^{N-1} \chi(-ut) \right) \left(\sum_{n \in \mathbb{Z}_p} \chi(r_n + un) \right) \\
&= \frac{N}{p} S(\mathbf{h}) + \frac{1}{p} \sum_{u=1}^{p-1} \left(\sum_{t=0}^{N-1} \chi(-ut) \right) \left(\sum_{n \in \mathbb{Z}_p} \chi(r_n + un) \right)
\end{aligned}$$

In the last line the summand for $u = 0$ was pulled out; the term $S(\mathbf{h})$ is the same as in the proof of Theorem 3.2. As we need an upper bound on $|S_N(\mathbf{h})|$, we apply the triangle inequation, yielding

$$|S_N(\mathbf{h})| \leq \frac{N}{p} |S(\mathbf{h})| + \frac{1}{p} \sum_{u=1}^{p-1} \left| \sum_{t=0}^{N-1} \chi(-ut) \right| \left| \sum_{n \in \mathbb{Z}_p} \chi(r_n + un) \right|, \tag{3.17}$$

and examine each of the terms on the right side.

We want to apply Lemma 3.4 on the rightmost term: For $1 \leq u \leq p-1$ we have, by the same argument as following (3.14)

$$\left| \sum_{n \in \mathbb{Z}_p} \chi(r_n + un) \right| \leq s + \left| \sum_{\substack{n \in \mathbb{Z}_p \\ R(n) \neq 0}} \chi\left(\frac{Q(n)}{R(n)} \right) \right|,$$

where Q/R is the rational function over $\mathbb{Z}_p$ given by

$$\frac{Q(x)}{R(x)} = \sum_{i=1}^{s} \frac{h_i}{a_i x + b_i} + ux \qquad \text{with} \quad R(x) = \prod_{i=1}^{s} (a_i x + b_i).$$

Once again, we claim that Q/R is not of the form $A^p - A$ with a rational function $A \in \overline{\mathbb{Z}}_p(x)$. From the definition of R and Q we have $\deg(R) = s$ and $\deg(Q) =$

$s+1$, as all the neither the a_i nor u can be zero. For if we had

$$\frac{Q}{R} = \left(\frac{K}{L}\right)^p - \frac{K}{L}$$

with polynomials K, L over $\overline{\mathbb{Z}}_p$ and $\gcd(K,L) = 1$, then the argument following (3.15) shows that L is a nonzero constant polynomial. Thus we have

$$Q = (e_1 K^p + e_2 K)R$$

for suitable $e_1, e_2 \in \overline{\mathbb{Z}}_p$ with $e_1, e_2 \neq 0$. Comparing the degrees of the polynomials in this equation we get $\deg(e_1K^p + e_2K) = 1$, which implies $\deg(K) \geq 1$, hence $\deg(e_1K^p + e_2K) = p\deg(K) > 1$. This contradiction proves that we can apply Lemma 3.4, yielding

$$\left|\sum_{n\in\mathbb{Z}_p} \chi(r_n + un)\right| \leq s(2p^{1/2}+1) \qquad \text{for } 1 \leq u \leq p-1. \tag{3.18}$$

Furthermore, by rewriting the sum over t using some elementary equivalences like $\chi(x) - 1 = e^{2\pi\sqrt{-1}x} - 1 = e^{\pi\sqrt{-1}x} \cdot (e^{\pi\sqrt{-1}x} - e^{-\pi\sqrt{-1}x}) = e^{\pi\sqrt{-1}x} \cdot 2\sqrt{-1}\sin\pi x$, we get

$$\begin{aligned}
\left|\sum_{t=0}^{N-1} \chi(-ut)\right| &= \left|\sum_{t=0}^{N-1} e^{2\pi\sqrt{-1}ut/p}\right| \\
&= \left|\sum_{t=0}^{N-1} \left(e^{2\pi\sqrt{-1}u/p}\right)^t\right| \\
&= \left|\frac{e^{2\pi\sqrt{-1}uN/p} - 1}{e^{2\pi\sqrt{-1}u/p} - 1}\right| \\
&= \left|\frac{e^{\pi\sqrt{-1}uN/p}\, 2\sqrt{-1}}{e^{\pi\sqrt{-1}u/p}\, 2\sqrt{-1}}\right| \cdot \left|\frac{\sin(\pi uN/p)}{\sin(\pi u/p)}\right| \\
&= \left|\frac{\sin(\pi uN/p)}{\sin(\pi u/p)}\right|. \qquad (3.19)
\end{aligned}$$

We now return to (3.17) to put the pieces together. Combining everything with an application of Lemma 3.3 we get

$$\begin{aligned}
&|S_N(\mathbf{h})| \leq \\
&\overset{(3.17)}{\leq} \quad \frac{N}{p}|S(\mathbf{h})| + \frac{1}{p}\sum_{u=1}^{p-1}\left|\sum_{t=0}^{N-1}\chi(-ut)\right|\left|\sum_{n\in\mathbb{Z}_p}\chi(r_n+un)\right| \\
&\overset{(3.18)}{\leq} \quad \frac{N}{p}|S(\mathbf{h})| + s(2p^{1/2}+1)\frac{1}{p}\sum_{u=1}^{p-1}\left|\sum_{t=0}^{N-1}\chi(-ut)\right|
\end{aligned}$$

$$\begin{aligned}
&\overset{(3.19)}{=} \frac{N}{p}|S(\mathbf{h})| + s(2p^{1/2}+1)\frac{1}{p}\sum_{u=1}^{p-1}\left|\frac{\sin(\pi uN/p)}{\sin(\pi u/p)}\right| \\
&\overset{\text{L. 3.3}}{<} \frac{N}{p}|S(\mathbf{h})| + s(2p^{1/2}+1)\left(\frac{4}{\pi^2}\,p\log p + (0.38)p + 0.64\right) \\
&\overset{(3.16)}{\leq} \frac{N}{p}\left((2s-2)p^{1/2}+s+1\right) + s(2p^{1/2}+1)\left(\frac{4}{\pi^2}\,p\log p + (0.38)p + 0.64\right).
\end{aligned}$$

for all $\mathbf{h} \in C_s^*(p)$, and thus we can apply Lemma 3.2 to obtain the desired upper bound on $D_N^{(s)}$. ∎

Lower Bounds

The theorems covering lower bounds are formulated in a different way. Instead of giving hard bounds for all EICG, these theorems state how many EICGs there must be exceeding a threshold value for the discrepancy. This kind of statement follows from the basic approach to the problem, namely combining upper bounds on exponential sums with their average values.

Lower bounds guarantee that the PRN are not perfectly equidistributed, they contain the irregularities found in "random" numbers, too.

The following three theorems are due to Eichenauer-Herrmann and Niederreiter [22].

Theorem 3.4 *Let $a_2 \in \mathbb{Z}_p^*$,$b_2 \in \mathbb{Z}_p$, and $c \in \mathbb{Z}_p \setminus \{b_2\overline{a_2}\}$ be fixed. Let $0 < t \leq \sqrt{p/(p-1)}$, and set*

$$A_p(t) := \frac{p^2 - (p-1)p\,t^2}{(2p^{1/2}+3)^2 - p\,t^2}.$$

Then there exist more than $A_p(t)$ values of $a_1 \in \mathbb{Z}_p^*$ such that for s-tuples from the corresponding parallel stream of EICG numbers with $b_1 = a_1 c$ and $s \geq 2$ we have

$$D_p^{(s)} \geq \frac{t}{2(\pi+2)}\,p^{-1/2}.$$

Proof: We rewrite the theorem in terms of E_N by using (3.12). Thus we have $\mathbf{d} = (d_1, d_2) = (\overline{a_1}, \overline{a_2})$, and $\mathbf{e} = (e_1, e_2) = (b_1\overline{a_1}, b_2\overline{a_2}) \in \mathbb{Z}_p^2$ with $e_1 \neq e_2$, and we need to show that there are more than $A_p(t)$ values for $d_1 \in \mathbb{Z}_p^*$ such that $|E(\chi; \mathbf{d}, \mathbf{e})| \geq tp^{1/2}$.

Now suppose there exist at most $A_p(t)$ values of $d_1 \in \mathbb{Z}_p^*$ with $|E(\chi; \mathbf{d}, \mathbf{e})| \geq tp^{1/2}$, i.e., there exist at least $(p-1) - A_p(t)$ values of d_1 with $|E(\chi; \mathbf{d}, \mathbf{e})| < tp^{1/2}$. From Lemma 3.6 (with $s = 2$) we know that $|E(\chi; \mathbf{d}, \mathbf{e})| \leq 2p^{1/2} + 3$ for every

$d_1 \in \mathbb{Z}_p^*$. Hence, observing that $d_1 = 0$ contributes nothing to the sum, we obtain

$$\sum_{d_1 \in \mathbb{Z}_p} |E(\chi; \mathbf{d}, \mathbf{e})|^2 = \sum_{d_1 \in \mathbb{Z}_p^*} |E(\chi; \mathbf{d}, \mathbf{e})|^2 < (p-1-A_p(t))t^2 p + A_p(t)(qp^{1/2}+3)^2 = p^2,$$

which contradicts Lemma 3.7 (with $s = 2$, $k = 1$, and $N = p$). ∎

Theorem 3.5 *Let $a_2 \in \mathbb{Z}_p^*$,$b_2 \in \mathbb{Z}_p$, $c \in \mathbb{Z}_p \setminus \{b_2\overline{a_2}\}$ and an integer N with*

$$\frac{1}{p}\left(2p^{1/2}\left(\frac{4}{\pi^2}\log p + 0.38 + \frac{0.64}{p}\right) + 2\right)^2 < N < p$$

be fixed and set

$$\begin{aligned} \tau_N &:= \frac{p}{p-1} - \frac{1}{N(p-1)}\left(2p^{1/2}\left(\frac{4}{\pi^2}\log p + 0.38 + \frac{0.64}{p}\right) + 2\right)^2 \\ A_N(t) &:= \frac{N(p-1)(\tau_N - t^2)}{\left(4p^{1/2}\left(\frac{4}{\pi^2}\log p + 0.38 + \frac{0.64}{p}\right) + \frac{N}{p}(2p^{1/2}+1) + 2\right)^2 - Nt^2} \end{aligned}$$

for $0 < t \le \sqrt{\tau_N}$. Then there exist more than $A_N(t)$ values of $a_1 \in \mathbb{Z}_p^*$ such that for s-tuples from the corresponding parallel stream of EICG numbers with $b_1 = a_1 c$ and $s \ge 2$ we have

$$D_N^{(s)} \ge \frac{t}{2(\pi+2)}\, N^{-1/2}.$$

Proof: The proof is analogous to the last one. The only real difference is the handling of $d_1 = 0$, where we need to apply Lemma 3.6 with $s = 1$. ∎

Using (3.13) instead of (3.12) we get a slightly different result. As the proof contains no new ideas, we omit it, too.

Theorem 3.6 *Let $c \in \mathbb{Z}_p$ and $1 \le N < p$ be fixed. For real numbers t fulfilling $0 < t \le \sqrt{(p-N)/(p-1)}$ set*

$$B_N(t) := \frac{N(p-N) - N(p-1)t^2}{\left(2p^{1/2}\left(\frac{4}{\pi^2}\log p + 0.38 + \frac{0.64}{p}\right) + \frac{N}{p} + 1\right)^2 - Nt^2}.$$

Then there exist more than $B_N(t)$ values of $a_1 \in \mathbb{Z}_p^*$ such that for s-tuples from the corresponding parallel stream of EICG numbers with $b_1 = a_1 c$ and $s \ge 2$ we have

$$D_N^{(s)} \ge \frac{t}{2}\, N^{-1/2}.$$

Restricting oneself to the ordinary serial test, i.e. is considering only overlapping s-tuples instead of vectors from parallel streams, it is possible to improve Theorem 3.6. See [22, Corr. 9] for details.

3.4 Other Results

For congruential generators modulo a prime p, the following definition, due to Niederreiter [69], specifies another criteria which can be used to classify PRNGs.

Definition 3.4 *For given $s \geq 1$, a congruential generator producing the sequence $y_0, y_1, \ldots \in \mathbb{Z}_p$ passes the* s-dimensional lattice test *if the vectors $\mathbf{y}_n - \mathbf{y}_0, n = 1, 2, \ldots$, span the s-dimensional vector space $\mathbb{Z}_p^s$, where*

$$\mathbf{y}_n = (y_n, y_{n+1}, \ldots, y_{n+s-1}) \in \mathbb{Z}_p^s \qquad \text{for } n = 0, 1, \ldots .$$

Theorem 3.7 *An EICG with modulus p passes the s-dimensional lattice test exactly for all $s \leq p - 2$. This is the optimal behaviour under this test.*

Proof: As explained in 1.3.4, one can visualize any congruential generator modulo p with period length p as a *permutation polynomial* mapping n to y_n. In the case of the EICG, this polynomial has degree $d = p - 2$ as we can write the the EICG formula as

$$y_n := \overline{an + b} = (an + b)^{p-2}$$

according to the theorem of Euler-Fermat. The rest follows immediately from a theorem by Eichenauer, Grothe, and Lehn [13], which can also be found in Niederreiter [69, Theorem 8.2]. ■

Chapter 4

Empirical Tests

As explained is section 2.3.2, testing a pseudorandom number generator is a tricky task. Even if one decides which properties one wants to test for, designing the test itself involves a fair amount of statistic knowledge as to how the test should be parameterized as well as how the results should be judged.

In this chapter, we will try to give a survey of empirical tests concerning the EICG carried out by various authors. Since it will be inappropriate to spend too much time on discussing all design decisions in detail, we will only describe the motivation, the test procedure, and the results. For more information we refer to the original authors.

Another compilation of empirical test results concerning inversive generators can be found in [20].

4.1 Digit Test

The Digit Test, due to Leeb[1] [57, 25, 60], tries to assess the distribution quality of a PRNG by looking at the g-adic representation of its output. If we only allow $g = 2^l$, one digit in base g corresponds to l binary digits, which we can obtain by cutting out l consecutive bits in the computer representation of the numbers. This is a very efficient procedure which can be done by simple bitwise `AND` and shift operations. Let's visualize this by an example, using $8 = 2^3$ as the base and selecting the second digit which is corresponds to the three bits starting at

[1]The author wants to thank Hannes Leeb for his help in writing this section, was well as for the Postscript graphics reprinted here.

position $k = 4$.

Decimal	Binary	Base 8	selected digit
0.5859375	0.100 101 10	0.454	5
0.82421875	0.110 100 11	0.646	4
0.21484375	0.001 101 11	0.156	5
0.765625	0.110 001 00	0.610	1
0.16015625	0.001 010 01	0.122	2
0.48046875	0.011 110 11	0.366	6

Ideally, we expect the last column to be uniformly distributed over all possible digits. Furthermore, we can assume that if the original numbers are uncorrelated, this will hold for the sequence of digits, too. On the other hand, if we can prove that something is wrong with the digit sequence, this does not shed a good light on the original numbers.

What have we gained by mapping numbers to digits ? Basically this mapping plays the role of the "bins" discussed on page 20 and 28. As discussed there, this mapping is used to make the problem manageable by drastically reducing the number of possible values for each number. Now we can easily count the occurrences of each different digit and perform correlation tests on them.

As a correlation test Leeb used the idea of the *serial test* (see p. 21) with *non-overlapping s-tuples*. To measure the distribution of the s-tuples, a χ^2 test was used; the resulting χ^2-value was called $t_1(s, k, l)$, the level-1 test statistic. This procedure was repeated 64 times, and all the χ^2 values were compared to their expected distribution by a two-sided Kolmogorov-Smirnov test, yielding $t_2(s, k, l)$, the level-2 test statistic on which we will focus in the graphics.

Let's summarize all the parameters which must be specified to turn the abstract idea of the Digit Test into a computer program.

- First of all, we have to specify the generator we want to test. Leeb used both a selection of popular linear congruential generators as well as inversive generators. In Table 4.2 the LCG are sorted according to their performance in the spectral test [9]; the smaller the value for $1/\nu_3$ the better is the lattice structure of the generator in dimension 3.
- Next, the number of bits to cut out is selected. This block-length l determines the base (2^l) of the digits to test. In his calculations, Leeb used $l \in \{1, 2, \ldots, 10\}$.
- Now that we know how many bits we want from each number, the next step is to determine *which* bits to cut out. Leeb termed this parameter the *block-start* k, which he chose from the set $\{1, 2, \ldots, 21\}$.

- We are now able to generate a sequence of digits (or bit-blocks) according to the parameters specified so far. The next step is to combine consecutive digits to vectors by forming non-overlapping s-tuples. Leeb chose to use dimensions s up to 10. As the dimension increases, the number of bins needed for the counting process grows exponentially; thus Leeb had to restrict the range of the block-length l with increasing s in order to keep the computation feasible.
- Next we need to determine how many s-tuples we should generate. The common rule of thumb for χ^2 tests is to make sure that the each of the 2^{sl} bins can expect at least 5 hits. As we expect equidistribution amongst the bins, we get a lower bound on how many tuples we should generate. Leeb decided to generate $6 \cdot 2^{sl}$ tuples, which were in turn generated using $s \cdot 6 \cdot 2^{sl}$ numbers from the PRNG.
- We have now finished the first level (t_1) of the Digit Test. For the second level (t_2), we need to repeat the above procedure with a distinct sample of PRN to be able to use the KS test. Leeb chose to use 64 repetitions which guarantees that the KS approximation is valid.

It should be clear by now that the hierarchical design of the complete testing procedure results in huge resource requirements both computationally as well as memory-wise to actually run the test. See Table 4.1 for the actual parameters used, and Table 4.2 for the generators tested. The computations were carried out using the pLab [56] PRNG testing framework, which is based on the author's generator library.

digit test parameters		
dimension s	block-start k	block-length l
1	$1, 5, 9, \ldots, 21$	$1, 2, \ldots, 10$
2	$1, 5, 9, \ldots, 21$	$1, 2, \ldots, 10$
3	$1, 5, 9, \ldots, 21$	$1, 2, \ldots, 7$
4	$1, 5, 9, \ldots, 21$	$1, 2, \ldots, 5$
5	$1, 5, 9, \ldots, 21$	$1, 2, \ldots, 4$
6	$1, 5, 9, \ldots, 21$	$1, 2, 3$
(7)	$(1), (5), (9), \ldots, (21)$	$(1), (2), (3)$
(8)	$(1), (5), (9), \ldots, (21)$	$(1), (2)$
$1, 2, \ldots, 10, (11)$	$1, 5, 9, \ldots, 29$	2

Table 4.1: Digit test parameters (parentheses indicate that only t_1 was computed)

generators			
nickname	generator	period	$1/\nu_3$
RANDU	$LCG(2^{31}, 65539, 0, 1)$	2^{29}	0.0920575
ANSI	$LCG(2^{31}, 1103515245, 12345, 12345)$	2^{31}	0.00132673
MINSTD	$LCG(2^{31}-1, 16807, 0, 1)$	$2^{31}-2$	0.00156518
FISH	$LCG(2^{31}-1, 950706376, 0, 1)$	$2^{31}-2$	0.000768506
EICG1	$EICG(2^{31}-1, 1, 0, 0)$	$2^{31}-1$	
ICG	$ICG(2^{31}-1, 1, 1, 0)$	$2^{31}-1$	

Table 4.2: Generators used in the Digit test

For the graphics, the t_1 value was transformed according to the expected χ^2 distribution to yield a value which should be asymptotically equidistributed. Thus all points in the graphics on the left hand side should vary freely between 0 and 1. Values close to 0 signify a distribution of the s-tuples which is much to well-balanced, whereas values close to 1 indicate gross irregularities. The right hand graphics depict the KS-statistic t_2, for which the critical region at the 1% level of significance is $[1.63, \infty)$. Any generator which features high values there (especially when reaching the cut-off point 2 in the graphics) fails in the test.

As Leeb in [57], we present here only a selection of the results, namely those for dimension $s = 3$. For each generator, the values of k and l were varied.

Interpretation: The digit test seems to be sensible to intrinsic properties of the LCG, as even the best one (FISH) fails the test for certain parameters. Leeb conjectures in [57] that the digit test is sensible on *grid structures* or *long-range correlation* as these are two features which are present in all LCG but are proven to be absent in inversive generators. Especially the lattice quality parameter $1/\nu_s$ seems to correlate with the digit test results. The better the lattice is (small values for $1/\nu_s$), the higher the values for l and k must be to uncover deficiencies in the generator.

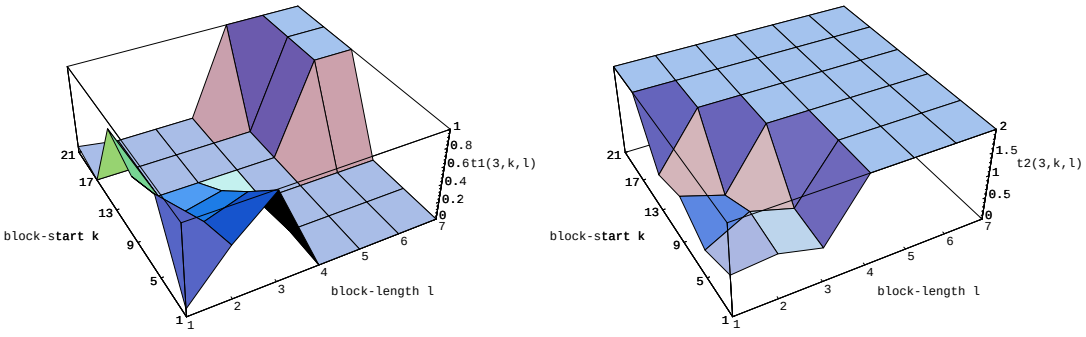


Figure 4.1: t_1 and t_2 for `RANDU` in dimension 3

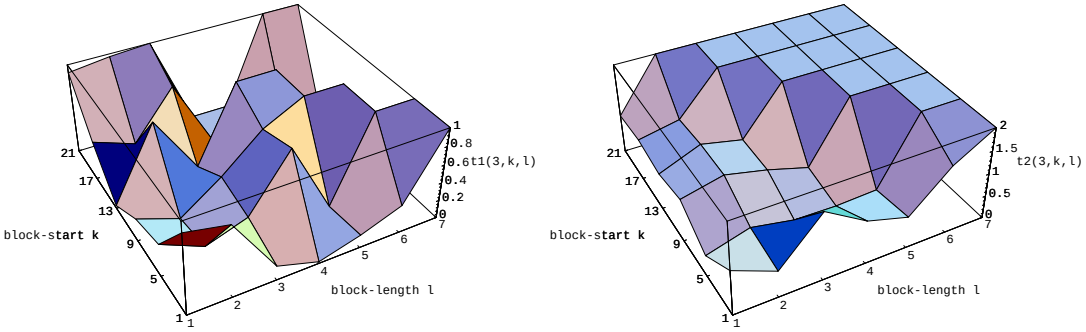


Figure 4.2: t_1 and t_2 for `ANSI` in dimension 3

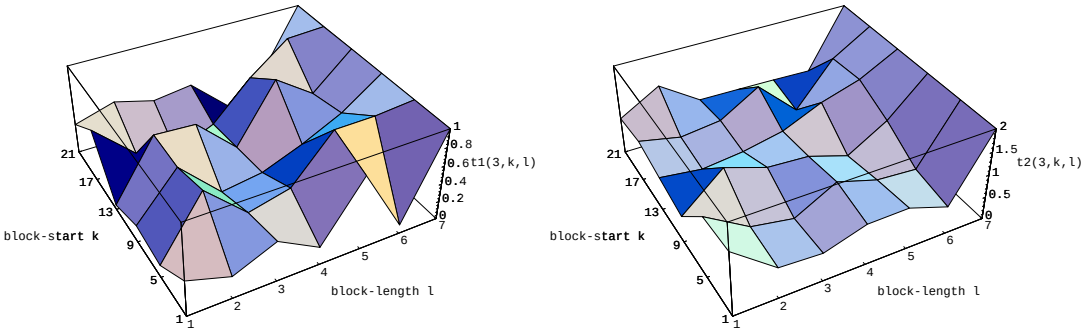


Figure 4.3: t_1 and t_2 for `MINSTD` in dimension 3

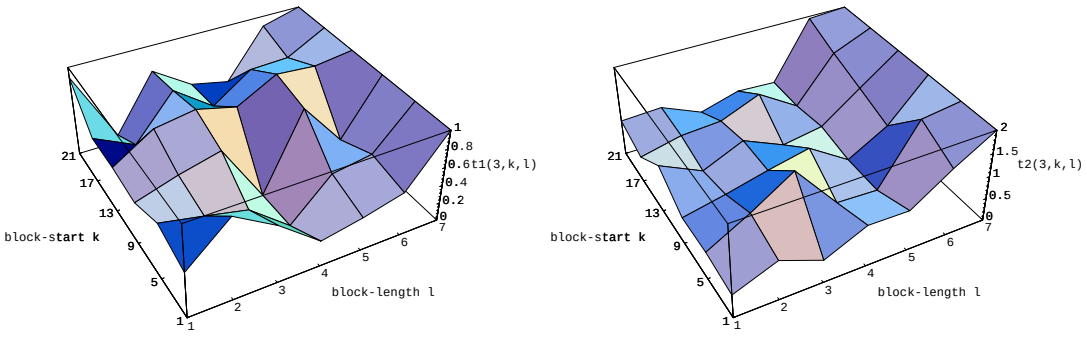


Figure 4.4: t_1 and t_2 for `FISH` in dimension 3

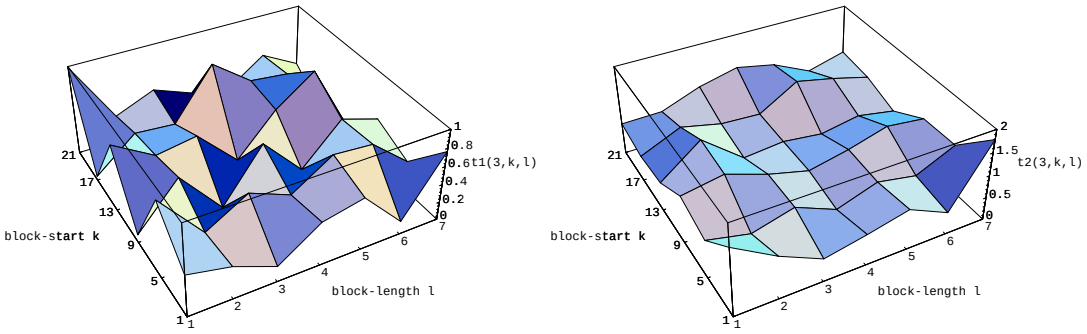


Figure 4.5: t_1 and t_2 for `EICG1` in dimension 3

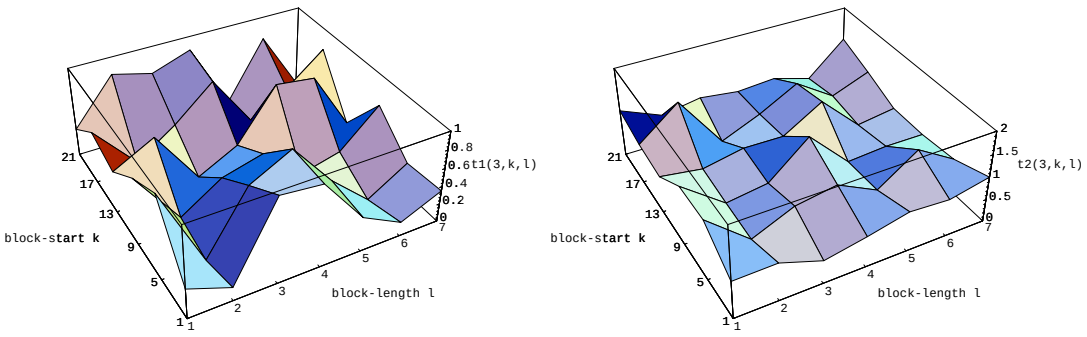


Figure 4.6: t_1 and t_2 for `ICG` in dimension 3

4.2 Overlapping Serial Test

The overlapping serial test, first proposed by Marsaglia [65] as the "M-tuple test", is in its basic setup quite similar to the digit test described above. The main difference is to use *overlapping* tuples. This modification is rather small in the implementation, but requires some statistical work to calculate the expected distribution as the tuples are *no longer independent*.

We will describe here the empirical tests done by Wegenkittl [84, 85, 60]. As the testing procedure was very similar to the digit test setup, we will only list the differences here. Whereas Leeb in the digit test used only one way to extract bits from the stream of pseudorandom numbers, Wegenkittl used two:

- Digit(*Start, Length*) is the same method as used in the digit test, namely cutting out *Length* bits starting at position *Start* from each number.
- BitStream(*Number, Length*) extracts more than one set of *Length* bits from each number, thus reducing the number of PRN needed to generate the tuples. This is achieved by cutting out *Number* times *Length* bits starting from the first bit. Furthermore, this method makes it feasible to test the numbers for correlations between the high-order and low-order bits.

This time these blocks of bits were used to generate variable number of tuples. Whereas in the digit test the sample size M was always tuned to the subsequent χ^2 test, Wegenkittl generated up to $M = 2^{26}$ tuples. From these tuples, a modified χ^2 statistic $t_1^{(o)}$ was computed, resulting in the test-statistic χ_o. Whereas the "normal χ^2 test statistic" does *not* converge to a χ^2 distribution for overlapping tuples due to the correlations, this modified one does (see [84, p. 57] for details). This procedure was repeated 32 times and the resulting empirical distribution of the χ_o values was compared to the theoretical one using a KS test.

The following graphics[2] depict the results for all the generators used in the digit test, as well as for `EICG7` which stands for $\mathrm{eicg}(2^{31}-1, 7, 0)$.

The left hand diagrams show the values for each of the 32 calculated $t_1^{(o)}$ test statistics as the lightness of each small rectangle. A white square signifies a low value for $t_1^{(o)}$, meaning perfect equidistribution of the tuples, whereas a black one signifies extreme deviations. The transformation $t_1^{(o)} \mapsto$ lightness was chosen in such way that each gray-scale level should be equally likely. Unfortunately, these diagrams are not available for all parameters.

In the right hand diagrams these 32 values were distilled into one single KS value representing the quality of their distribution. The critical region at the 1% level of significance is in this case $[1.58, \infty)$.

[2] We like to thank Stefan Wegenkittl for providing the Postscript graphics, as well as for his helpful comments on this section.

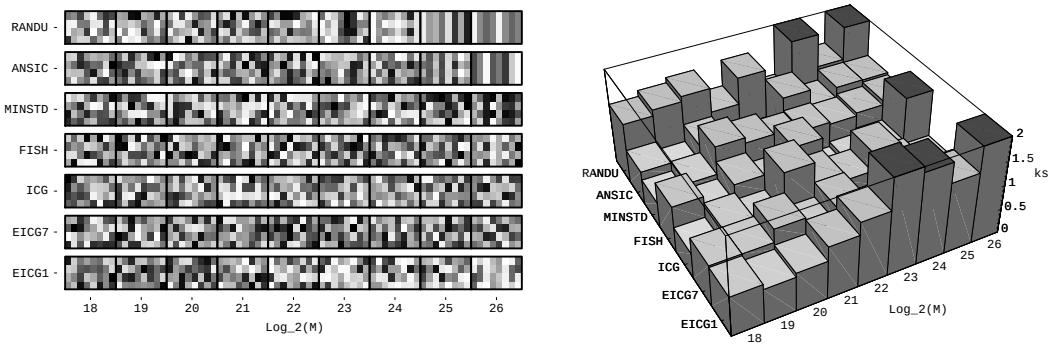


Figure 4.7: Dimension 2: Digit(1,4)

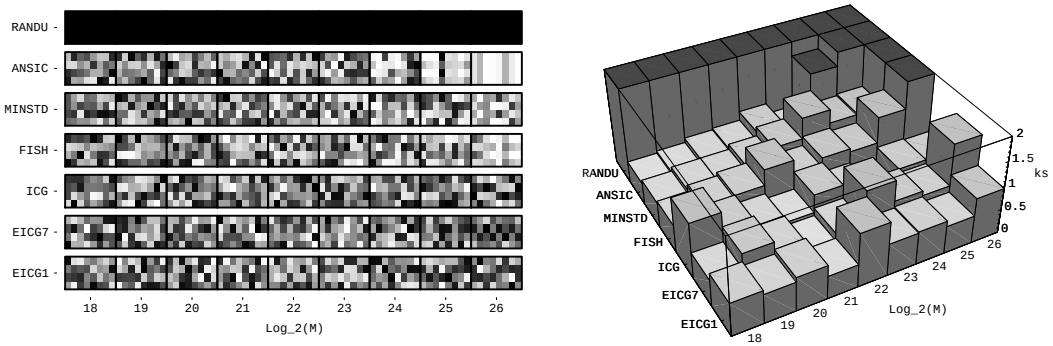


Figure 4.8: Dimension 3: Digit(1,4)

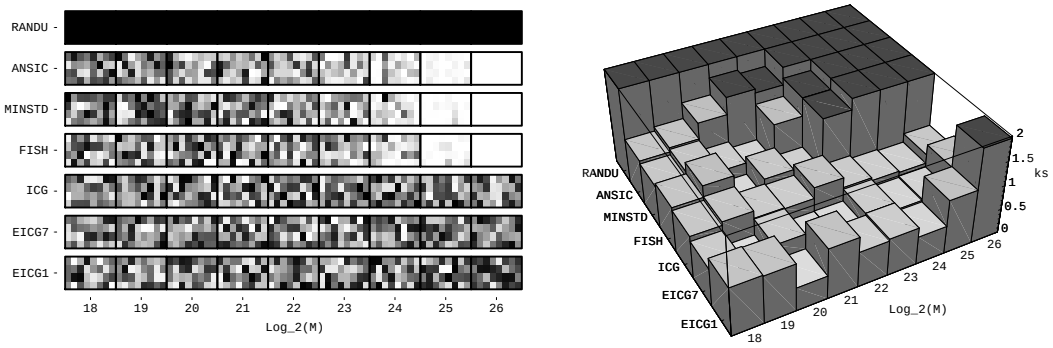


Figure 4.9: Dimension 4: Digit(1,4)

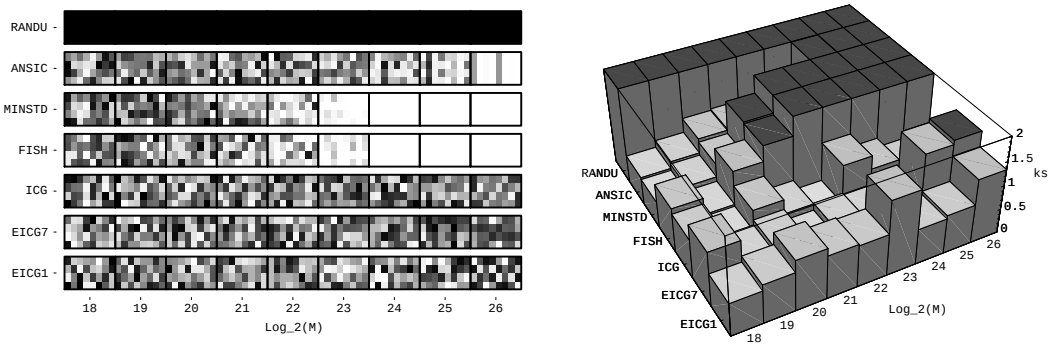


Figure 4.10: Dimension 5: Digit(1,4)

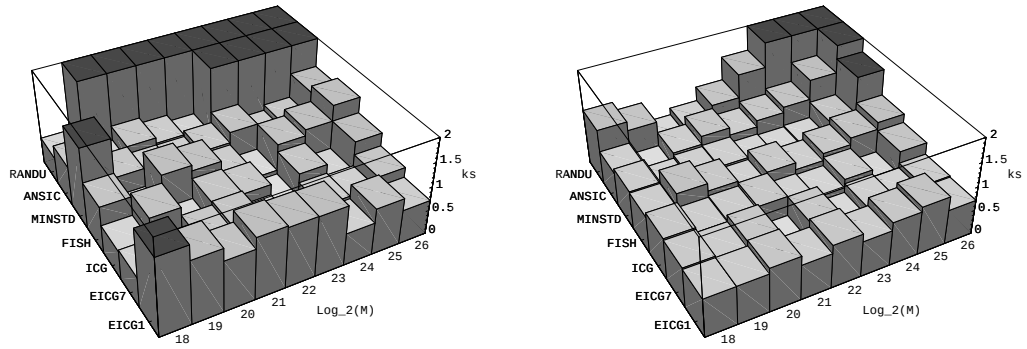


Figure 4.11: BitStream(4,4): Dimensions 2 and 3

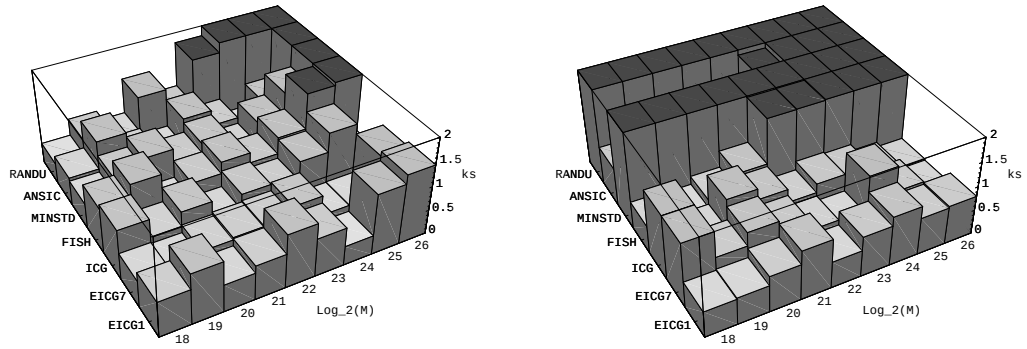


Figure 4.12: BitStream(4,4): Dimensions 4 and 5

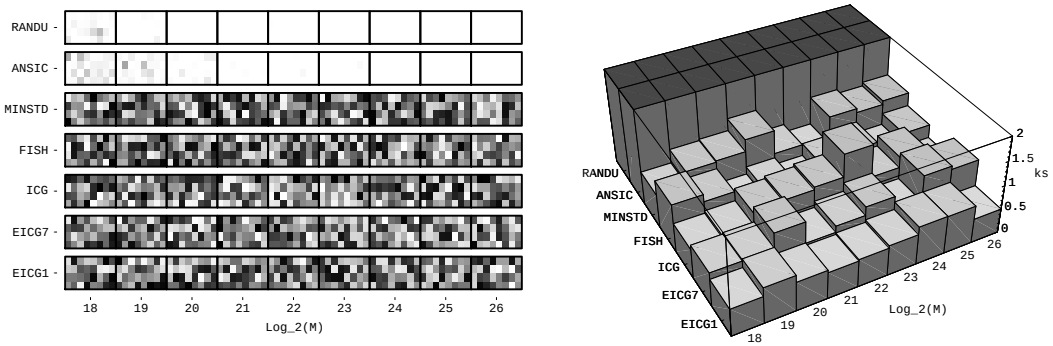


Figure 4.13: Dimension 2: BitStream(6,4)

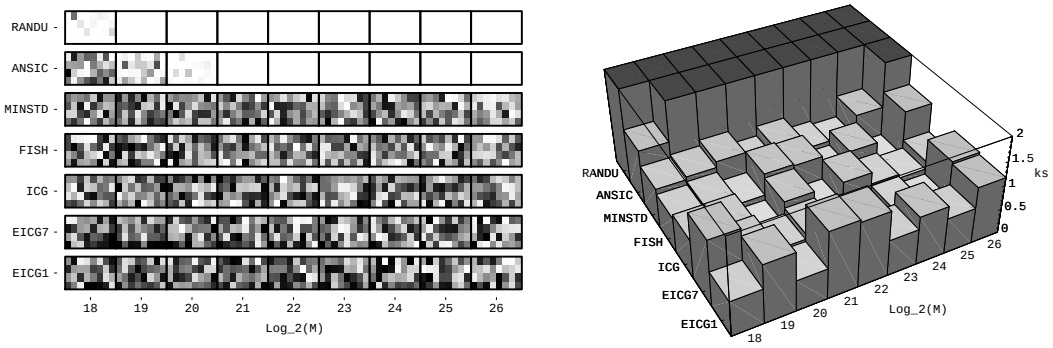


Figure 4.14: Dimension 3: BitStream(6,4)

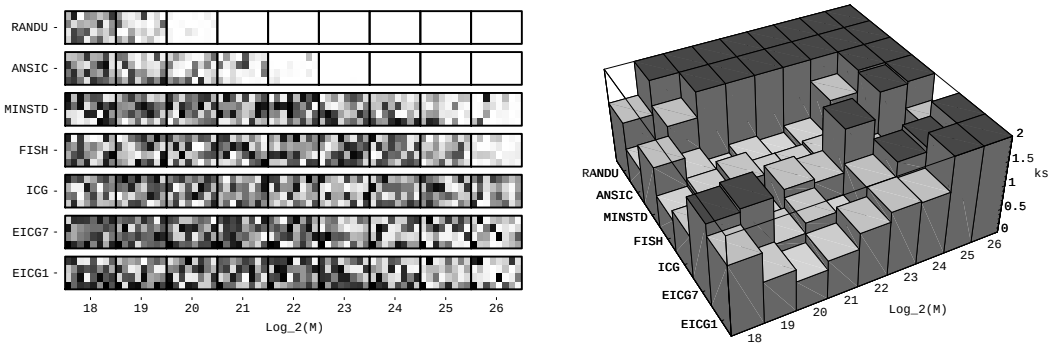


Figure 4.15: Dimension 4: BitStream(6,4)

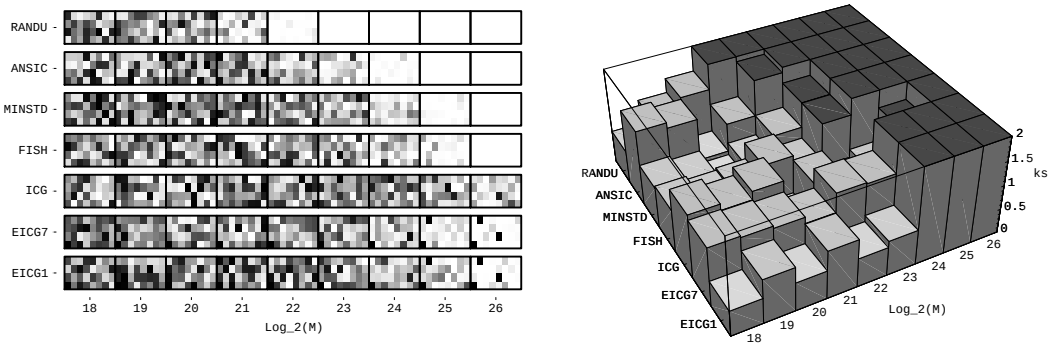


Figure 4.16: Dimension 5: BitStream(6,4)

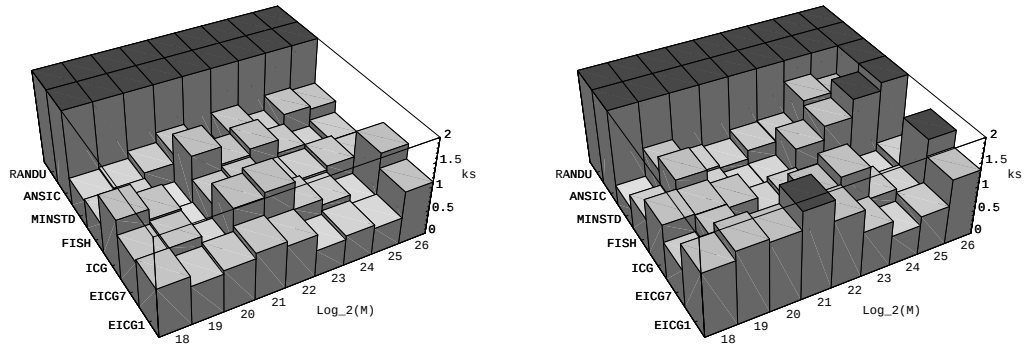


Figure 4.17: BitStream(8,4): Dimensions 2 and 3

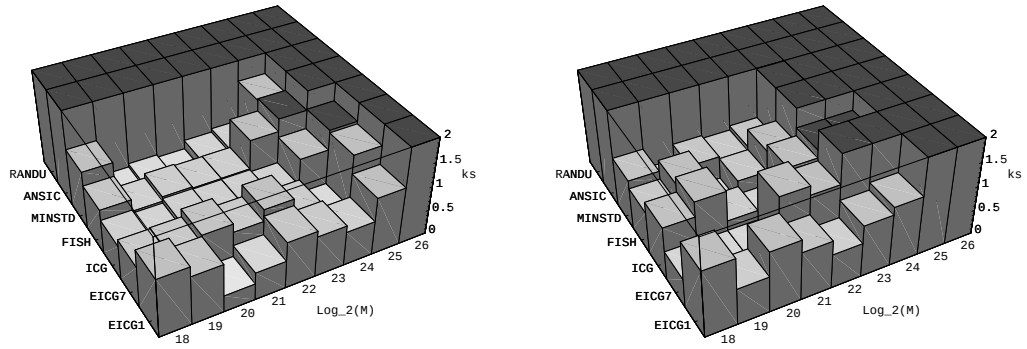


Figure 4.18: BitStream(8,4): Dimensions 4 and 5

Interpretation: Like the Digit Test, the overlapping serial test does uncover deficiencies in the linear generators. Whereas the setup in the Digit Test focused on the number of bits you can take from a number while still getting good distributions, Wegenkittl turned his attention to the number of tuples one can generate before the PRNG fails. The results are basically the same: even the best LCG has a limited *load-capability*; if you start to do heady-duty computations using many PRN you must be careful not to run into the breaking point of the LCG. The traditionally rule of thumb concerning which percentage of the period length one can safely use (up to $\sqrt{\text{period}}$) seems to be sound for the LCG.

The inversive generators are not perfect either, even they tend to fail the test at some point. But their load-capability is much better. Even when taking a high number of samples, their distribution quality decreases quite slowly.

4.3 Run Test

A completely different kind of empirical test is the run test. The basic idea behind this test is to check if the occurrences of *runs* conforms to its expected value. There is a variety of different ways to implement this idea, but we will only describe the idea and Entacher's implementation [23].

First of all, what do we mean by "runs" ? In a binary context, that is sequence consisting only of two symbols, a run is defined as a subsequence consisting only of one symbol. For example, consider the sequence

$$(\underbrace{1,}\ \underbrace{0,0,0,}\underbrace{1,1,1,1,}\underbrace{0,0,}\ \underbrace{1,}\ \underbrace{0,}\ \underbrace{1,}\ \underbrace{0,0,0,}\underbrace{1,1,1,1})$$

in which the brace indicate the runs.

As the common pseudorandom sequences are far from being binary, one has to transform them first. A straight forward way of doing this is

$$x_n \mapsto \operatorname{sign}(x_{n+1} - x_n),$$

which is binary as $x_{n+1} \neq x_n$ for PRNG we consider. A run of 1s of length k corresponds to a monotonically increasing subsequence of length $k+1$ in the original sequence called a "run up". The distribution properties of these runs in a sequence of "really random numbers" is well known.

According to Wolfowitz [62] (see also Knuth [49, p. 68]) Entacher constructed an asymptotically χ^2-distributed test statistic U_r which involves counts for ascending runs up to length r. The calculation done by Entacher involved evaluating U_6 100 times for each generator and testing this empirical distribution against the expected one using a KS test.

We will focus on Entacher's results concerning the behaviour of LCGs and EICGs. The list of generators tested include the previously defined generators `FISH`, `ANSI`, `MINSTD`, and `RANDU`, as well as two other LCGs with a bad lattice structure: `LCG5` $= \mathrm{lcg}(2^{31}, 2^{31} - 3, 0, 1)$ and `LCG6` $= \mathrm{lcg}(2^{31} - 1, 2^{21} + 1, 0, 1)$. As examples for the EICG Entacher used $\mathrm{eicg}(2^{31} - 1, a, 0, 0)$ with parameter $a \in \{2^5, 2^{10}, 2^{15}, 2^{20}, 2^{25}, 2^{30}\}$ which he labeled `EICG1` to `EICG6`.

The following figures[3] depict the results; The sample size was varied between 2^{12} and 2^{24} (the labels are $\log_2(N)$), the height of each square shows the resulting KS statistic. If the square is coloured dark gray, then the KS statistic exceeds the critical value for the significance level 0.01.

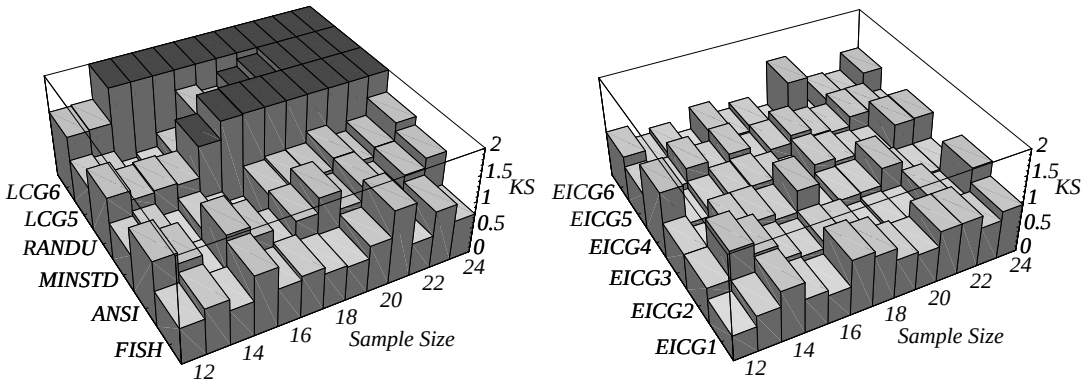


Figure 4.19: LCGs and EICGs in the Run Test

Interpretation: The run test seems to be able to distinguish good LCGs from bad ones. All basically randomly chosen EICG pass the test without any problems, showing again that the EICG is not sensitive to the parameter selection.

Another conclusion from this test is that subsequences taken by a leap frog technique are save when using EICGs, whereas such sequences taken from an LCG may exhibit a bad lattice and can fail the run test. See an upcoming paper from Entacher [23] for details.

[3]Once again we want to thank the author of the original paper, Karl Entacher, for his support and the graphics files.

4.4 Weighted Spectral Test

As mentioned on page 43, the weighted spectral test is a promising new approach to assess the quality of pseudorandom numbers, see Hellekalek [39, 40], Hellekalek and Niederreiter [45], and Hellekalek and Leeb [43]. One numerical realization of the weighted spectral test is the *diaphony*, see Hellekalek and Niederreiter [45].

Both the diaphony as well as the classic spectral test [9] approach the point set from a Fourier point of view, looking for any disturbances in the spectrum. If the point set has lattice structure (as in the case of LCGs), the first wavelength which yields a non-zero Fourier coefficient corresponds to the largest distance between hyperplanes. Where the classic spectral test targets just this wavelength, the diaphony tries to include more information, namely a weighted sum over all possible wavelengths. High wavelengths (which correspond to low frequencies) in the point set indicate a large scale imbalance, whereas high frequencies target fine structures in the set. As these fine structures are unavoidable at the certain point, higher frequencies are considered less important and thus will contribute little to the diaphony.

So basically the s-dimensional diaphony is a weighted sum over the correlation coefficients of the point set, which are basically the Weyl sums S_N we encountered in Section 3.3, see [40] for details.

There are close ties between the discrepancy and the diaphony. One can bound one in terms of the other (see Stegbuchner [78]), one can interpret it, too, as integration error (see James, Hoogland and Kleiss [46]), and it is possible to derive Lemmata similar to those in section 3.3.2 (see [40]).

Whereas it is virtually impossible to do any reasonable empirical studies with discrepancy due to its computational complexity of $\mathcal{O}(N^s)$, the diaphony only needs $\mathcal{O}(s \cdot N^2)$ steps to calculate it. Thus it is possible to conduct empirical test using this figure of merit. In the following we will describe the results obtained by Hellekalek [39].

The test procedure was as follows: For a given generator, for a given dimension s, and a given sample size N, the diaphony F_N^2 was evaluated for 20 samples of non-overlapping s-tuples. As the expected value for F_N^2 equals $1/N$, Hellekalek multiplied the diaphony by N to get the same expected value for all sample sizes, see Leeb [59] for the theoretical distribution.

In the graphics[4], the abscissa shows the sample size as $\log_2 N$ and the ordinate the average value over the 20 samples. For each generator from the now familiar set (see Table 4.2) the calculation was done in dimensions 2 ($\triangle$), 3 ($\circ$), 4 ($\star$), 5 ($\Diamond$), and dimension 6 ($|$).

[4]We like to thank Peter Hellekalek for the permission to use his graphics, as well as for providing the mathematical background.

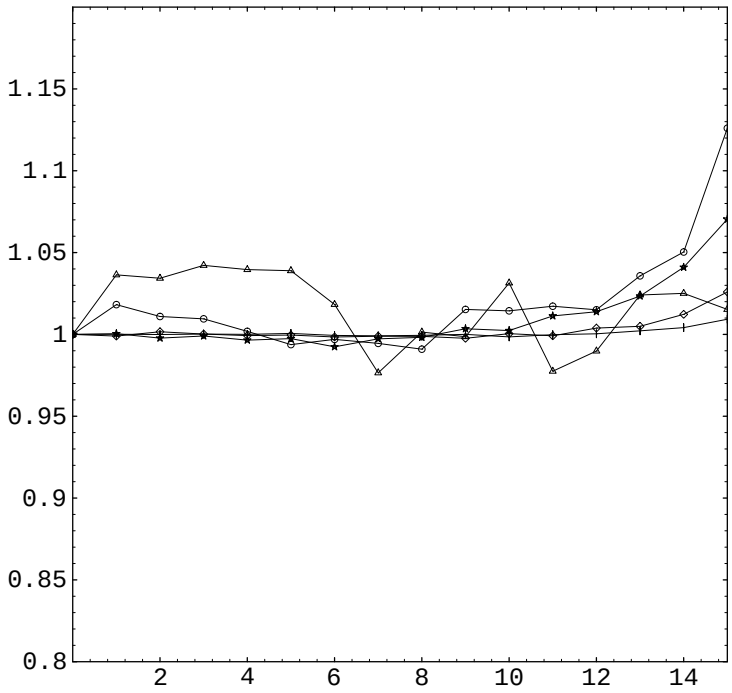


Figure 4.20: Diaphony for `RANDU`

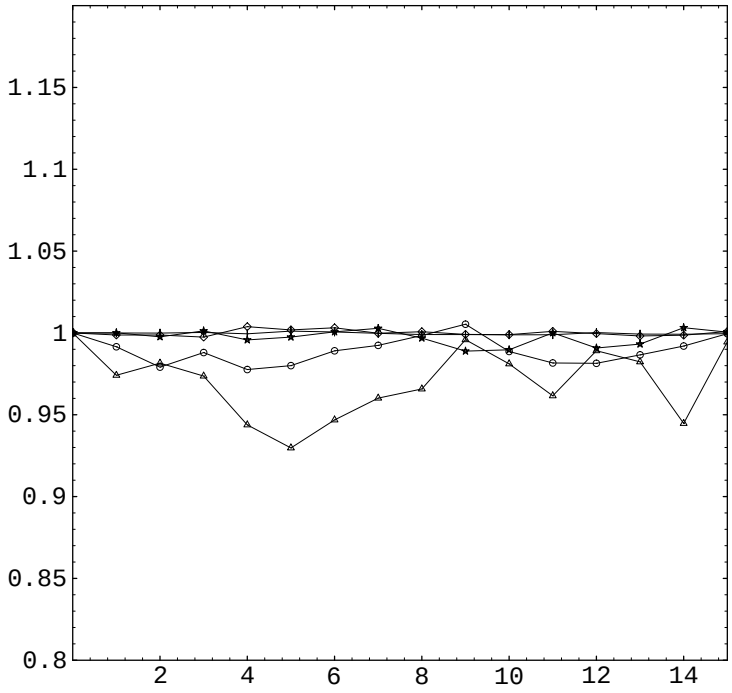


Figure 4.21: Diaphony for `ANSI`

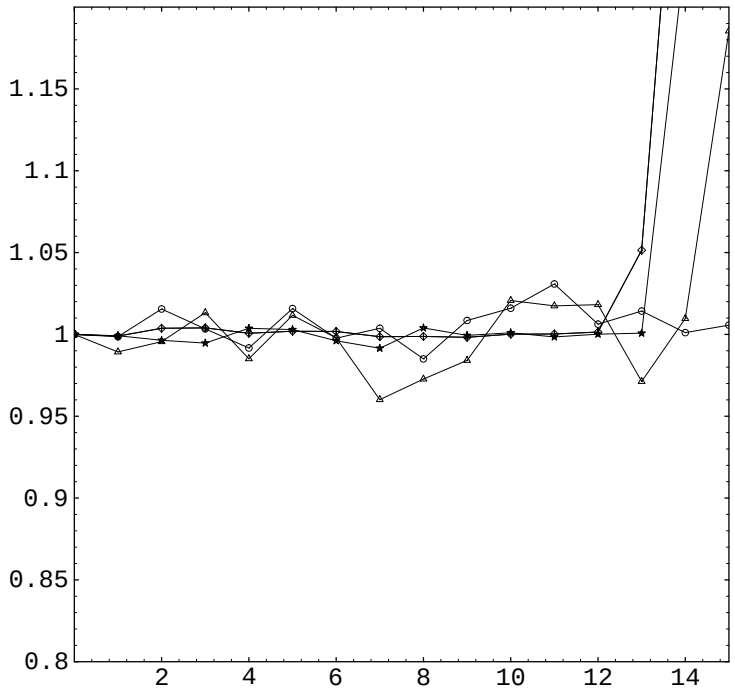


Figure 4.22: Diaphony for `MINSTD`

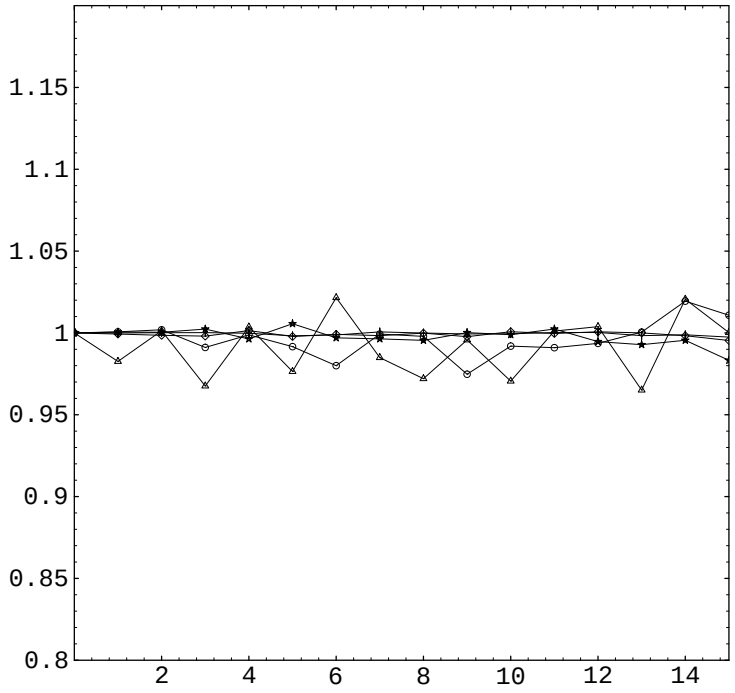


Figure 4.23: Diaphony for `FISH`

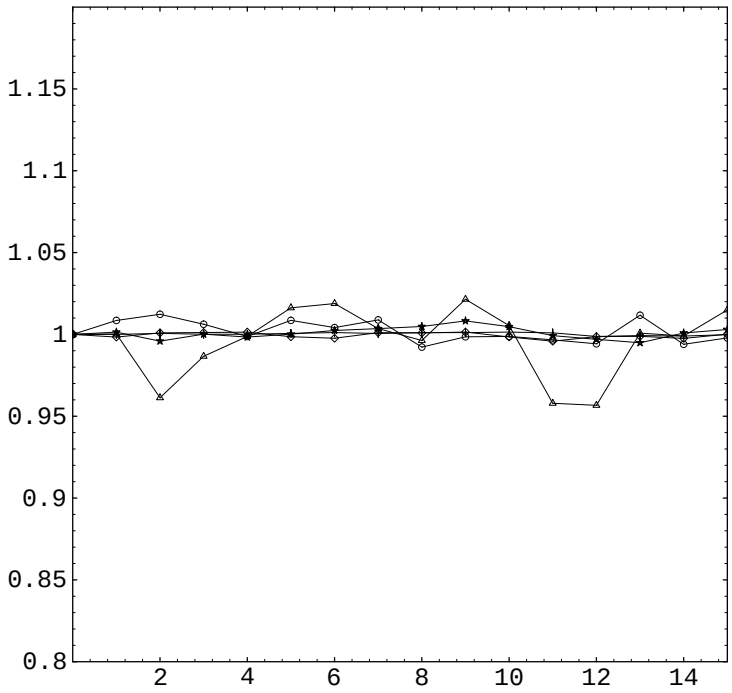


Figure 4.24: Diaphony for `ICG`

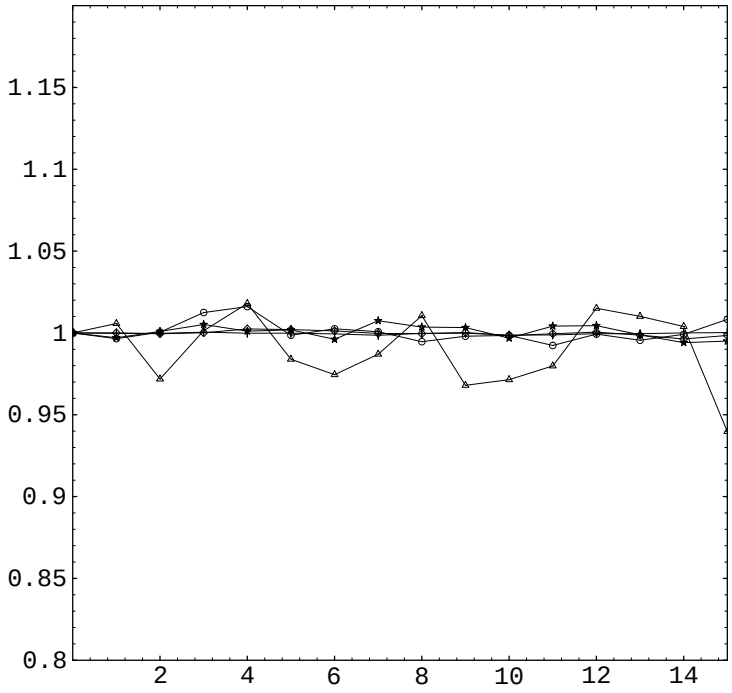


Figure 4.25: Diaphony for `EICG1`

4.5 Other Results

Frank Härtel [36] implemented an impressive array of PRNG and compared their performance under a variety of statistical tests. Unfortunately he used only one EICG (eicg$(1000081, 240318, 197, 0)$), whose modulus ($p < 2^{20}$) is far too small to be able to compete with generators featuring a period length of 2^{32}. We will therefore not elaborate on his results.

Both Eichenauer-Herrmann and L'Ecuyer presented results of empirical test concerning the *minimal distance* between vectors of PRN on the MC&QCM'96 conference in Salzburg. Although the results have not been published yet, one can summarize them as follows: due to their lattice structure, LCGs are not able to simulate the correct behaviour of the test statistic whereas EICGs with the same period length pass this test with flying colors. I refer to the upcoming proceedings for details.

Chapter 5

Implementation

This section discusses the implementation of the EICG pseudorandom number generator using a standard procedural programming language. We will use `C` syntax for the code printed here.

The algorithms presented here were used to write a generic and portable PRNG library which implements not only the EICG, but other congruential generators, too. This library (written in `ANSI C`) is available on the Internet from the pLab WWW server at `http://random.mat.sbg.ac.at/`.

5.1 Overview

If we look at the definition of the EICG (see p. 1.3.4), we see that generating the numbers is a two step procedure. First we have to compute the $y_n := \overline{a(n_0 + n) + b}$, a calculation operating in the finite field $\mathbb{Z}_p$ which can be done by standard integer calculation modulo p. The second step is the scaling operation $x_n := y_n/p$, which we will implement as a straight forward floating point division.

We will thus focus on the first step, which includes the following three operations

1. Inversion modulo p
2. Multiplication modulo p
3. Addition modulo p.

How they are best implemented depends on how numbers are represented in the computer. We use integer arithmetic based on the native integer format of the computer. Most current workstations use 32-bit integers which can hold numbers

from -2^{31} to $2^{31}-1$. This limits the choice of the modulus to values smaller than 2^{31}, a common choice is the Mersenne prime $2^{31}-1$. Large values for p give the resulting PRN a fine resolution, but one has to pay for this with an increase in calculation time. If this resolution, as well as the period length it implies, are not adequate, one can use the technique of combining generators (see p. 16), to generate even better pseudorandom numbers.

5.2 Modular Inversion

First we turn to the problem of modular inversion (denoted throughout this text by overlining ($\overline{a}$) the operand) which is defined as

$$\overline{a} = \begin{cases} a^{-1} & \text{for } a \neq 0 \\ 0 & \text{for } a = 0, \end{cases}$$

where a^{-1} is the uniquely defined element of $\mathbb{Z}_p^* = \mathbb{Z}_p \backslash \{0\}$ such that $a\overline{a} \bmod p = 1$.

The special case $a = 0$ is easily handled, what remains is to find a^{-1} for $a \neq 0$. There are two different ways to compute the inverse, one of them is to utilize the fact that

$$a^{\varphi(p)} \equiv 1 \pmod{p}$$

by the well-known theorem of Euler–Fermat, and thus $a^{\varphi(p)-1} \equiv \overline{a} \pmod{p}$ holds. In our case here the modulus is prime, thus $\varphi(p)$, Euler's totient function, is equal to $p-1$, which gives us

$$\overline{a} \equiv a^{p-2} \pmod{p}.$$

Evaluating $a^b \pmod{p}$ is a well-known exercise in computational number theory (e.g. in the RSA cryptosystem). It can be solved in logarithmic time [8, p. 829], but intermediate results exceed the domain $[-p, p]$. This fact renders an implementation difficult because of the limited integers available on a computer.

A different approach is to use an extended version of Euclid's algorithm. This algorithm is usually used to calculate the greatest common divisor of two numbers, but it can also be used to calculate the integers x and y which fulfill the linear diophantic equation $ax + by = \gcd(a, b)$. By substituting p for b and observing that $\gcd(a, p) = 1$, one gets

$$ax + py = 1,$$

which can be rewritten as

$$ax \equiv 1 \pmod{p}.$$

This algorithm to calculate x and y is based on the following recursion: The division with remainder $a = qb + r$ is used to calculate q and r. If we can find

```
/*
 * Extended Euclid's Algorithm, Recursive version.
 *
 * From: "Algorithms" by Corman, Leiserson, & Rivest [8]. page 812
 *
 * Input:
 *     a,b    Two integers
 *
 * Output:
 *     int d,x,y     which satisfy gcd(a,b) = d = ax + by                    10
 *
 */
void rec_eeuclid(int a,int b,int *d,s_int *x,s_int *y)
{
int dbar,xbar,ybar;

if (0 == b)
        {
        *d = a;
        *x = 1;                                                             20
        *y = 0;
        return;
        }

rec_eeuclid(b,a%b,&dbar,&xbar,&ybar);

*d = dbar;
*x = ybar;
*y = xbar - (a/b) * ybar;
}                                                                           30
```

Figure 5.1: Euclid's algorithm, recursive version.

x' and y' which fulfill $x'b + y'r = 1$, then $x = y'$ and $y = x' - qy'$ will satisfy $xa + yb = 1$. Since $b < a$ and $r < b$ the question how to find the x' and y' will lead to a trivial case.

Figure 5.1 shows a straight-forward recursive implementation of the extended Euclid's algorithm. Figure 5.2 demonstrates how the modular inversion can be based on `rec_eeuclid`.

This recursive implementation is not very efficient due to the overhead caused by the repeated function calls. Although `rec_eeuclid` is not end-recursive, it is possible to rewrite it as an iterative function [49]. Figure 5.3 is a `C` implementation of the modular inversion using this method. A further optimization [76, p. 521] is to unroll the loop twice to avoid unnecessary swapping of the variables in each iteration.

```
/*
 * Modular Inversion. Based on Euclid's Algorithm
 *
 * Input:
 *     a,p    Two int, gcd(a,p) should be 1 !
 *
 * Output:
 *     int a' which satisfies:
 *           a' * a = 1    (for a != 0)
 *           a'     = 0    (for a == 0)                                    10
 *
 */
int inverse(int a,int p)
{
int gcd,inv,temp;

if (a == 0)
      return(0);

rec_eeuclid(a,p,&gcd,&inv,&temp);                                          20

if (inv < 0)
      inv += p;

if (gcd != 1)
      fputs("inverse: Can't invert !", stderr);

return(inv);
}
```

Figure 5.2: Modular inversion based on `rec_eeuclid`

Gordon [34] describes a modification which uses shift operations to avoid multiplications and divisions. This does pay on certain computers, for example on SPARC, R4000, or Alpha AXP based systems this approach is faster. On the other hand, the division on the i486 is comparatively fast, thus the original version runs faster there.

The number of recursive calls in Euclid's algorithm is of the order $O(\lg b)$, see [8, p. 810]. An equivalent statement is, that the arguments of these calls decrease exponentially. Another way to put this is that `rec_eeuclid` will need about the same number of recursive calls to get from $b \approx 16384$ to $b \approx 128$ as it needs from $b \approx 128$ to the end of the recursion.

This observation leads to another technique to speed up the computation of the multiplicative inverse: For all a, b smaller than some threshold we use a table of precomputed values for x and y instead of continuing the recursion. This way

some recursive calls can be avoided.

Threshold	Memory used	Avg. Steps	Std.dev.	Time
No Table	-	17.57	3.27	43.76 s
16	256 Bytes	16.72	3.10	45.40 s
32	1 KByte	16.11	3.04	43.84 s
64	4 KByte	15.52	2.98	42.32 s
128	16 KByte	14.93	2.92	41.03 s
256	64 KByte	14.35	2.86	39.71 s
512	512 KByte	13.77	2.80	39.31 s

Table 5.1: Average number of recursive calls in Euclid's Algorithm.

To test the advantages of this approach, we compared it with the optimized iterative implementation. Table 5.1 lists the timings of my test case, that is calculating the multiplicative inverses of 2147483 uniformly distributed numbers modulo $2^{31} - 1$.

As long as the threshold is not larger than 256, a byte is enough to hold an element of the table. A threshold of 512 forces the program to resort to 16-bit integers which doubles the memory requirements. It is hard to make a general statement which threshold is best, too much depends on the cache size, memory access speed, and on memory available. We have set the default to 256, which seems to result in a reasonable tradeoff between memory and speed.

Our experience has shown that there is no single "best" algorithm, too much depends on the relative execution speed of various elementary operations. Thus our implementation includes three different algorithms as well as a profiling program which can be used to select the one which is running fastest on the user's computer.

5.3 Modular Multiplication

The problem of evaluating $a \cdot n \pmod{p}$ lies in the limited range of the integers available in common programming languages. The intermediate result an of the straightforward implementation is very likely not to fit in machine size integers, so one has to devise an algorithm to calculate $an \pmod{p}$ in which all intermediate results are representable on a b-bit computer.

One approach is the following algorithm due to Bratley, Fox, and Schrage [4, Sec. 6.5.2] which can compute $an \pmod{p}$ if $a^2 < p$. The idea is to factor the modulus, but since this is not possible with primes, one has to deal with remainders, too. Let

$$p = aq + r$$

where

$$q = p \operatorname{div} a \qquad \text{and} \qquad r = p \bmod a.$$

Then one can rewrite $an \pmod p$ as

$$\begin{aligned} an \pmod p &= an - p\,(an \operatorname{div} p) \\ &= an - p\,(n \operatorname{div} q) + p\,\underbrace{(n \operatorname{div} q - an \operatorname{div} p)}_{\delta(n)} \\ &= an - aq\,(n \operatorname{div} q) - r\,(n \operatorname{div} q) + p\,\delta(n) \\ &= a\,(n - q\,(n \operatorname{div} q)) - r\,(n \operatorname{div} q) + p\,\delta(n) \\ &= a\,(n \bmod q) - r\,(n \operatorname{div} q) + p\,\delta(n) \\ &= \gamma(n) + p\,\delta(n) \end{aligned}$$

If $r < q$, which is a direct consequence of $a^2 < p$, evaluating $\gamma(n)$ does not pose a numerical problem, for

$$a\,(n \bmod q) < aq \le p$$

and

$$r\,(n \operatorname{div} q) < q\,(n \operatorname{div} q) \le n \le p,$$

and thus $|\gamma(n)| \in \{0, \ldots, p-1\}$. Since $an \pmod p \in \{0, \ldots, p-1\}$ evaluating $\delta(n)$ is unnecessary, because $\delta(n) = 0$ iff $\gamma(n) \ge 0$ and $\delta(n) = 1$ iff $\gamma(n) < 0$. This leads to the following algorithm:

$$an \pmod p = \begin{cases} \gamma(n) & \text{if } \gamma(n) \ge 0 \\ \gamma(n) + p & \text{otherwise} \end{cases}$$

q and r can be precomputed, calculating $(n \operatorname{div} q)$ and $(n \bmod q)$ requires on some computers only one instruction, so this is a very efficient algorithm.

For the usual choice of $2^{31} - 1$ for p the above can be applied for all $a < 2^{15}$. If p is smaller than that, then the limitation that all intermediate results should be between $-p$ and p is unnecessarily tight. On a b-bit computer all integers between -2^{b-1} and 2^{b-1} (exclusive) are representable. If we loosen the restriction on a from $a^2 < p$ to $a^2 < 2^{b-2}$, the term $r\,(n \operatorname{div} q)$ is no longer bounded by p. But it can be shown that 2^{b-1} is an upper bound:

$$\begin{aligned} r\,(n \operatorname{div} q) &< r\,(p \operatorname{div} q) \\ &= r\,((aq + r) \operatorname{div} q) \\ &\le r\,(a + r \operatorname{div} q) \\ &\le a\,(a + a \operatorname{div} q) \\ &\le 2a^2 \\ &< 2^{b-1} \end{aligned}$$

We can now conclude that $-2^{b-1} < \gamma(n) < p$ and $an \pmod p = \gamma(n) \pmod p$. Thus one possible algorithm is this:

1. Calculate q and r
2. $x = a\,(n \bmod q) - r\,(n \operatorname{div} q)$
3. while$(x < 0)$ $\quad x = x + p$

The while loop can execute at most $\lceil 2^{b-1}/p \rceil$ times. L'Ecuyer and Côté [55] observed that in the average case very few iterations are executed, thus this algorithm is efficient.

If a is not restricted in any way (except of course by the requirement $a < p$) one can use *decomposition* to reduce the multiplication to cases for which we already have solutions. This can be achieved by writing a in base 2^d, where $d = (b-2)/2$ (usually 15 on current 32-bit computers):

$$a = a_0 + a_1 2^d + a_2 2^{2d}$$

with $0 \le a_0, a_1 < 2^d$, and $a_2 \in \{0, 1\}$. Then

$$\begin{aligned} an \pmod p &= \\ &= (a_0 n) \bmod p + ((a_1 n) \bmod p) 2^d \bmod p + ((a_2 n) \bmod p) 2^{2d} \bmod p \\ &= ((((a_2 2^d n) \bmod p + (a_1 n) \bmod p) \bmod p) 2^d \bmod p + a_0 n \bmod p) \bmod p. \end{aligned}$$

In all four products modulo p one of the factors is bounded by 2^d, so the previously discussed algorithm can be applied. Figure 5.4 shows a C implementation of this method. It is used whenever it is not possible to resort to a simpler algorithm.

5.4 Modular Addition

Modular addition is simple to solve, though it is not trivial. A straightforward implementation might look like this:

1. $x = a + b$
2. if $(x \ge p)$ then $x = x - p$

This is correct, as long as $a + b$ is still fully representable with the data type used. Assuming that the usual signed integer type is used, an overflow would occur if $a + b$ is negative. It cannot happen that $a + b$ is positive in spite of an overflow, since a and b are both smaller than $2^{b-1} - 1$ (assuming a b-bit computer) and $(2^{b-1} - 1) + 1$ wraps to -2^{b-1}. If we detect an overflow, subtracting p will bring the result back into the interval $[0, p - 1]$. Thus this

1. $x = a + b$
2. if $((x < 0)$ or $(x \ge p))$ then $x = x - p$

is a correct implementation.

```
/*
 * Modular Inversion. Direct implementation using Euclid's alg. [34]
 *
 * Input:
 *     a,p    Two integers
 *
 * Output:
 *     Modular inverse of a modulo p. (0 if a == 0)
 *
 */                                                                    10
int inverse2(int a,int p)
{
int q,d,u,v,inv,t;

if (a <= 1) return(a);

d = p; inv = 0; v = 1; u = a;
do
{
        q = d / u;                                                     20
        t = d % u;      /* On my Linux box this is faster than d - q*u */
        d = u;
        u = t;
        t = inv - q*v;
        inv = v;
        v = t;
} while (u != 0);

if (inv < 0 ) inv += p;
                                                                       30
if (1 != d)
        fprintf(stderr,"Can't invert %d modulo %d !\n",a,p);

return(inv);
}
```

Figure 5.3: Modular inversion based on Euclid's algorithm (iterative version).

```
/*
 * Modular Multiplication: Decomposition method (from L'Ecuyer & Cote [55])
 *
 * Input:
 *     a,n,p Three integers. a,n < p.
 *
 * Output:
 *     (a*n) mod p
 *
 */                                                              10
int mult_mod(int a,int n,int p)
{
int H,a0,a1,q,qh,rh,k,x;

H = 32768;   /* 2 ^ 15 */

if (a < H)
      { a0 = a; x = 0; }
 else
      {                                                          20
      a1 = a / H ; a0 = a - H * a1;
      qh = p / H ; rh = p - H * qh;
      if ( a1 >= H )
            {
            a1 = a1 -H; k = n / qh;
            x = H*(n- k*qh) - k * rh;
            while(x < 0) x += p;
            }
       else
            x = 0;                                               30

      if ( a1 != 0 )
            {
            q = p / a1; k = n / q;
            x = x -k*(p-a1*q); if (x>0) x-=p;
            x = x + a1*(n-k*q); while (x<0) x+=p;
            }
      k = x / qh ; x = H*(x - k*qh) - k*rh;
      while (x<0) x+=p;
      }                                                          40
if (a0 != 0)
      {
      q = p / a0; k = n /q;
      x = x -k*(p-a0*q); if (x>0) x-=p;
      x = x + a0*(n-k*q); while (x<0) x+=p;
      }
return(x);
}
```

Figure 5.4: Decomposition method for modular multiplication.

Chapter 6

Summary

Let us quickly summarize the main points of this thesis.

- **The application defines the quality criteria for the PRNG.** There is no such thing as the perfect ("one size fits all") pseudorandom number generator; one should always consider the application when choosing a generator.
- **Testing a PRNG is a tricky task.** While testing can increase the confidence in the generator, it is often not possible to construct a test which targets the same properties in the number as will be relevant in the application.
- **Do not rely on a single family of PRNG !** As a consequence of the previous two remarks, one should never trust the result of a simulation without verification runs using a completely different PRNG.
- **The definition of the EICG.**

  Let p be a (large) prime and $a, b, n_0 \in \mathbb{Z}_p$. The explicit inversive congruential generator *(abbreviated as "EICG") with parameters p, a, b, and n_0 defines a sequence $(y_n)_{n\geq 0}$ in $\mathbb{Z}_p$ by*

  $$y_n := \overline{a \cdot (n_0 + n) + b} \qquad (n \geq 0)$$

  and a sequence $\mathrm{eicg}(p, a, b, n_0) = (x_n)_{n\geq 0}$ *of pseudorandom numbers in* $[0, 1[$ *by*

  $$x_n := \frac{y_n}{p} \qquad (n \geq 0),$$

  where $\overline{c}$ denotes the multiplicative inverse modulo p extended by $\overline{0} := 0$.
- **The choice of parameters is easy for the EICG.** As long as $a \neq 0$, the period length will always be p. There are *no* theoretical results indicating

that some parameters might result in bad distribution properties. This is a major advantage over the LCG, where every set of parameters must be extensively screened using the spectral test.

- **EICGs with the same modulus are closely related.** These relation include shifting the sequence (different n_0 or b), and taking every k-th number (different value for a).

- **Taking subsequences from EICGs is safe.** As a consequence of the last two statements, the EICG is perfectly suited to generate streams of PRN. This can be done by taking either every k-th number or starting at a different position in the stream. In both cases, the individual streams are guaranteed to be uncorrelated.

- **Tuples of EICG numbers show strong non-linear properties.** See Theorems 3.1 and 3.7 for details.

- **The order of magnitude of the discrepancy of N s-tuples of EICG numbers is close to the optimal value.** For true random number we have an order of magnitude between $N^{-1/2}$ and $N^{-1/2}\sqrt{\log\log N}$. Examining tuples formed using the full period of the EICG we get $D_p^{(s)} = \mathcal{O}(p^{-1/2}(\log p)^s)$ as an upper bound, and an existence statement which implies that this bound is the best possible up to the logarithmic factor.

  For parts of the period we get similar results: as an upper bound we have $D_N^{(s)} = \mathcal{O}(N^{-1}p^{1/2}(\log p)^{s+1})$, and we can show the existence of EICGs with $D_N^{(s)} \geq cN^{-1/2}$ for some constant c.

- **EICGs perform well in empirical tests.** In all comparisons to linear generators with about the same period length the inversive generators have clearly proved their superiority.

- **The EICG is not particularly difficult to implement.** Short and efficient algorithms for all steps involved have been published. A portable implementation (ANSI-C) is available from the author.

- **The EICG algorithm runs at a reasonable speed.** The multiplicative inversions takes about $\log p$ steps; the EICG is thus considerably slower than the LCG. Whether this difference is noticeable in the overall computation time depends heavily on the amount of processing done in the simulation problem itself.

- **Combining EICGs is safe.** Using compound techniques it easy to achieve long periods; excellent properties of the resulting numbers are guaranteed [17, 18].

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